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ABSTRACT

This paper presents finite dimensional approximations for linear retarded functional differential equations by use of discontinuous piecewise linear functions. The approximation scheme is applied to optimal control problems, when a quadratic cost integral has to be minimized subject to the controlled retarded system. It is shown that the approximate optimal feedback operators converge to the true ones both in case the cost integral ranges over a finite time interval as well as in the case it ranges over an infinite time interval. The arguments in the latter case rely on the fact that the piecewise linear approximations to stable systems are stable in a uniform sense. This feature is established using a vector-component stability criterion in the state space $\mathbb{R}^n \times L^2$ and the favorable eigenvalue behavior of the piecewise linear approximations.

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1. INTRODUCTION

Given $\phi^0 \in \mathbb{R}^n$ and $\phi^1: [-h, 0] \rightarrow \mathbb{R}^n$ consider the retarded functional differential equation with constant coefficients

$$\dot{x}(t) = \sum_{k=0}^p A_k x(t-h_k) + \int_{-h}^0 A_{01}(s)x(t+s)ds, \quad t \geq 0$$

$$x(0) = \phi^0, \quad x = \phi^1 \quad \text{in } L^2(-h, 0; \mathbb{R}^n).$$

An equivalent abstract Cauchy problem $\dot{z}(t) = Az(t)$, $t \geq 0$; $z(0) = (\phi^0, \phi^1)$ in the space $M^2 = \mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n)$ generates a strongly continuous semigroup. Approximations are constructed by restricting the problem to finite dimensional subspaces $Z^N = \mathbb{R}^n \times Y^N \subseteq M^2$, defining appropriate generators A^N on Z^N .

Banks and Burns [1] used subspaces Y^N which consist of functions that are piecewise constant on the delay interval $[-h, 0]$. This is the well-known averaging approximation scheme. As an extension, Burns and Cliff [5] enlarged the subspaces to piecewise linear functions. In both papers, the approximating generators were constructed by forward difference methods. In [3] the approximations were obtained by projections onto subspaces of continuous splines being contained in the domain of A . Then Kappel and Salamon [14] introduced δ -type operators in order to define generators for a spline scheme whose adjoint semigroups converge strongly. These δ -type operators are specially constructed to approximate the differential operator A at the discrete delays, where the splines may be discontinuous, as are the functions in the domain of A^* .

In this paper, a new scheme is presented employing again subspaces of orthogonal piecewise linear functions as in [5], but using δ -type operators for the construction of the approximating generators. These operators are needed at each meshpoint, where the subspace functions may be discontinuous. In fact, the number of discontinuities increases with the order of the approximation, in contrast to the spline case. The resulting generators are completely different from those given in [5].

An application of the approximation schemes is the optimal control problem, when an integral ranging over $[0, T)$, quadratic in the trajectory and in the control, is to be minimized subject to the controlled delay equation. It was shown by J. S. Gibson [9] that, if $T < \infty$, the strong convergence of the semigroups and their adjoints yields convergence in norm of the optimal feedback operators. In this present work, strong convergence of the semigroups and their adjoints is proved using the Trotter-Kato Theorem as in [1], [9]. Thereby, it is not necessary to assume absolute continuity of A_{01} , as did the proofs in [12], [13], [14].

In the case of the so-called infinite time horizon $T = \infty$, Gibson's approach relies on the assumption that a stable system is approximated by systems that are stable in a uniform sense. For the averaging scheme, this stability preservation property was established in [19], and in [12] for the Legendre-tau methods. In contrast, the spline schemes do not have this quality (see [14]). This is due to extraneous eigenvalues close to the imaginary axis. It is shown below that the eigenvalues of the present scheme converge to those of the delay equation and that exponential stability of our approximations is dominated by their \mathbb{R}^n -components. Thus, uniform preservation of exponential stability is proved with decay rates arbitrarily close to the decay rate of the hereditary system.

The matrices corresponding to the piecewise linear functions are banded and sparse, in contrast to the Legendre and spline methods. While the Legendre schemes [11], [13] exhibit high accuracy even for low order of approximation, the numerical efficiency of the present scheme is about the same as that of the first order splines in [14], and superior to the averaging methods [1] as well as those in [5].

Preliminarily, Section 2 collects some facts on the semigroup generated by the uncontrolled system and on the linear quadratic hereditary control problem (see [6], [8], [9]). Section 3 presents an approximation framework suited to Chapter 4, where the piecewise linear scheme is developed. In Section 4.1, the δ -type operators are defined and the projections onto the subspaces of piecewise linear functions are investigated. In Section 4.2, the approximating generators and their adjoints are constructed and convergence results for the finite time horizon problem are proved. In 4.3, the matrix representations are given and an \mathbb{R}^n -component stability criterion is established. Section 4.4 investigates the eigenvalue behavior of the approximate systems when the order of approximation increases. In Section 4.5, the uniform stability preservation is proved. Finally, in 4.6 there is a brief discussion of the numerical tools needed for the implementation of the scheme on a computer and the results of three examples are tabulated.

2. THE LINEAR QUADRATIC OPTIMAL CONTROL PROBLEM FOR HEREDITARY SYSTEMS

We are concerned with the linear retarded functional differential equation

$$(2.1;1) \quad \dot{x}(t) = Lx_t + B_0 u(t), \quad t \geq 0$$

in \mathbb{R}^n , where for some fixed finite delay $h > 0$, $x_t: [-h,0] \rightarrow \mathbb{R}^n$ is defined by $x_t(s) = x(t+s)$, B_0 is a real $n \times m$ -matrix and $u(t) \in \mathbb{R}^m$. The bounded linear functional $L: C(-h,0; \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is given by

$$L\phi^1 = \sum_{k=0}^p A_k \phi^1(-h_k) + \int_{-h}^0 A_{01}(s) \phi^1(s) ds,$$

with $0 = h_0 < \dots < h_p = h$, $A_k \in \mathbb{R}^{n \times n}$, $k = 0, \dots, p$ and $A_{01} \in L^2(-h,0; \mathbb{R}^{n \times n})$.

Given $\phi = (\phi^0, \phi^1) \in M^2 = \mathbb{R}^n \times L^2(-h,0; \mathbb{R}^n)$ and $u \in L^2_{loc}(0, \infty; \mathbb{R}^m)$, there exists a unique solution $x(t; \phi, u)$, which is absolutely continuous with L^2 -derivative on every interval $[0, T]$, and satisfies (2.1;1) for almost all $t \geq 0$, and the initial condition

$$(2.1;2) \quad x(0; \phi, u) = \phi^0, \quad x(\cdot; \phi, u) = \phi^1 \quad \text{in } L^2(-h,0; \mathbb{R}^n).$$

Defining the state at time t by

$$(2.2) \quad z(t; \phi, u) = (x(t; \phi, u), x_t(\phi, u)) \in M^2,$$

system (2.1) is converted to an abstract Cauchy problem in M^2 , which is a Hilbert space with the inner product $\langle (\phi^0, \phi^1), (\psi^0, \psi^1) \rangle = \phi^{0T} \psi^0 + \langle \phi^1, \psi^1 \rangle_2$.

Let $S(t)$, $t \geq 0$ be the C_0 -semigroup corresponding to the free motion of (2.1), i.e.,

$$S(t)\phi = (x(t; \phi, 0), x_t(\phi, 0)), \quad t \geq 0, \quad \phi \in M^2$$

and define the input operator $B: \mathbb{R}^n \rightarrow M^2$ by

$$Bu = (B_0 u, 0), \quad u \in \mathbb{R}^n.$$

Then the evolution of $z(t; \phi, u)$ in time is governed by the variation of constants formula

$$(2.3) \quad z(t; \phi, u) = S(t)\phi + \int_0^t S(t-s)Bu(s)ds, \quad t \geq 0.$$

The infinitesimal generator A of $S(\cdot)$ is given by

$$\text{dom } A = \{\phi \in M^2 \mid \phi^1 \in W^{1,2}(-h, 0; \mathbb{R}^n), \phi^1(0) = \phi^0\},$$

$$A\phi = (L\phi^1, \dot{\phi}^1),$$

where $W^{j,p}(a,b; \mathbb{R}^n)$ denotes the space of \mathbb{R}^n -valued absolutely continuous functions on $[a,b]$ possessing $j-1$ continuous derivatives and j -th derivatives that are in $L^p(a,b; \mathbb{R}^n)$. The function $z(t; \phi, u)$ in (2.3) is a mild solution of the abstract system

$$(\dot{\sum}) \quad \dot{z}(t) = Az(t) + Bu(t), \quad t \geq 0, \quad z(0) = \phi.$$

Weighting the past history by the step function

$$g(s) = p - k + 1, \quad s \in [-h_k, -h_{k-1}), \quad k = 1, \dots, p,$$

we have an equivalent norm corresponding to the inner product

$$\langle \phi, \psi \rangle_g = \phi^0 T \psi^0 + \int_{-h}^0 \phi^1(s) T \psi^1(s) g(s) ds, \quad \phi, \psi \in M^2.$$

If

$$(2.4) \quad \omega \geq \frac{p}{2} + |A_0| + \frac{1}{2} \sum_{k=1}^p |A_k|^2 + \|A_{01}\|_{L^2(-h,0; \mathbb{R}^{n \times n})}$$

the operator $A - \omega I$ is dissipative in M_g^2 , that is

$$\langle A\phi, \phi \rangle_g \leq \omega \|\phi\|_g^2, \quad \phi \in \text{dom } A.$$

Therefore, by the Lumer-Phillips Theorem, there exists a constant $M > 1$ such that

$$(2.5) \quad \|S(t)\|_{M^2} \leq M e^{\omega t}, \quad t \geq 0.$$

The M^2 -adjoint of A generates the M^2 -adjoint semigroup $S(t)^*$, $t \geq 0$ and is given by

$$\text{dom } A^* = \left\{ \phi \in M^2 \mid \phi^1 + \sum_{k=1}^{p-1} A_k^T \phi^0 \chi_{[-h, -h_k)} \in W^{1,2}(-h, 0; \mathbb{R}^n), \quad \phi^1(-h) = A_p^T \phi^0 \right\},$$

$$(A^* \phi)^0 = \phi^1(0) + A_0^T \phi^0,$$

$$(A^* \phi)^1 = A_{01}^T \phi^0 - \frac{d}{ds} \left(\phi^1 + \sum_{k=1}^{p-1} A_k^T \phi^0 \chi_{[-h, -h_k)} \right).$$

χ_I denotes the characteristic function of the interval I . The optimal control problem on a finite interval is: given $0 < T < \infty$ and $\phi \in M^2$ find the control $u \in L^2(0, T; \mathbb{R}^n)$ that minimizes the cost functional

$$J(u, \phi, T) = x(T; \phi, u)^T G_0 x(T; \phi, u) + \int_0^T (x(t; \phi, u)^T W_0 x(t; \phi, u) + u(t)^T R u(t)) dt,$$

where $G_0 = G_0^T$ and $W_0 = W_0^T$ are nonnegative matrices and $R = R^T$ is positive definite. Defining the operators $G: M^2 \rightarrow M^2$ by $G\phi = (G_0 \phi^0, 0)$ and $W: M^2 \rightarrow M^2$ by $W\phi = (W_0 \phi^0, 0)$ we have

$$(2.6) \quad \begin{aligned} J(u, \phi, T) = & \langle z(T; \phi, u), Gz(T; \phi, u) \rangle + \\ & + \int_0^T (\langle z(t; \phi, u), Wz(t; \phi, u) \rangle + u(t)^T R u(t)) dt. \end{aligned}$$

The solution to the linear quadratic control problem (2.3), (2.6) in the Hilbert space M^2 is based on the Riccati differential equation

$$(2.7) \quad \begin{aligned} \frac{d}{dt} \langle \psi, \Pi(t)\phi \rangle + \langle A\psi, \Pi(t)\phi \rangle + \langle \Pi(t)\psi, A\phi \rangle - \\ - \langle \Pi(t)\psi, BR^{-1}B^* \Pi(t)\phi \rangle + \langle \psi, W\phi \rangle = 0, \quad \text{for } \phi, \psi \in \text{dom } A, \quad 0 \leq t \leq T, \end{aligned}$$

$$\Pi(T) = G.$$

There exists a unique, strongly continuous family of nonnegative selfadjoint operators $\Pi(t)$, $t \geq 0$, satisfying (2.7). The optimal control \bar{u} is given in feedback form

$$(2.8) \quad \bar{u}(t) = -R^{-1}B^* \Pi(t)\bar{z}(t), \quad 0 \leq t \leq T,$$

where $\bar{z}(t)$ is the mild solution of

$$\dot{\bar{z}}(t) = (A - BR^{-1}B^* \Pi(t))\bar{z}(t), \quad \bar{z}(0) = \phi,$$

i.e.,

$$\bar{z}(t) = S(t)\phi - \int_0^t S(t-s)BR^{-1}B^* \Pi(s)\bar{z}(s)ds.$$

The optimal cost is $J(\bar{u}, \phi, T) = \langle \phi, \Pi(0)\phi \rangle$. With respect to the product structure of M^2 , we can write

$$\Pi(t) = \begin{bmatrix} \Pi_{00}(t) & \Pi_{01}(t) \\ \Pi_{10}(t) & \Pi_{11}(t) \end{bmatrix}, \quad 0 \leq t \leq T.$$

$\Pi_{00}(t): \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\Pi_{11}(t): L^2(-h, 0; \mathbb{R}^n) \rightarrow L^2(-h, 0; \mathbb{R}^n)$ are selfadjoint and nonnegative. Since $\Pi_{10}(t): \mathbb{R}^n \rightarrow L^2(-h, 0; \mathbb{R}^n)$, it has a representation by a matrix valued function $\Pi_{10}(t, \cdot) \in L^2(-h, 0; \mathbb{R}^{n \times n})$, i.e.,

$(\Pi_{10}(t)\xi)(s) = \Pi_{10}(t, s)\xi$, $0 \leq t \leq T$, $\xi \in \mathbb{R}^n$. Using this notation, we write for $\phi^1 \in L^2(-h, 0; \mathbb{R}^n)$

$$\Pi_{10}^*(t)\phi^1 = \Pi_{01}(t)\phi^1 = \int_{-h}^0 \Pi_{10}^T(t, s)\phi^1(s)ds.$$

Since $B^*\phi = B_0^T\phi^0$, $\phi \in M^2$, the feedback law can be written in terms of the delay system:

$$\bar{u}(t) = -R^{-1}B_0^T[\Pi_{00}(t)\bar{x}(t) + \int_{-h}^0 \Pi_{10}^T(t,s)\bar{x}(t+s)ds],$$

$\bar{x}(t)$ being the solution of the closed loop system

$$x(t) = (A_0 - B_0R^{-1}B_0^T\Pi_{00}(t))x(t) + \sum_{k=1}^p A_k x(t-h_k) + \\ + \int_{-h}^0 (A_{01}(s) - B_0R^{-1}B_0^T\Pi_{10}^T(t,s))x(t+s)ds, \quad 0 \leq t \leq T.$$

We also consider the infinite time horizon problem, that is the minimization of $J(u,\phi)$ given by (2.6) with $G \equiv 0$ and $T = \infty$, and assume that the system (2.1;1) is stabilizable, i.e., $A - BK$ generates an exponentially stable semigroup for some linear bounded operator $K: M^2 \rightarrow \mathbb{R}^n$. Then there exists a nonnegative, selfadjoint operator $\Pi \in L(M^2)$ that maps $\text{dom } A$ into $\text{dom } A^*$ and satisfies the algebraic Riccati equation

$$(2.9) \quad A^* \Pi \phi - \Pi A \phi - \Pi B R^{-1} B^* \Pi \phi + W \phi = 0, \quad \phi \in \text{dom } A.$$

If, in addition, system (2.1;1) and W_0 have the property, that any admissible control drives the state to zero, that is $J(u,\phi) < \infty$ implies $z(t;\phi,u) \rightarrow 0$, as $t \rightarrow \infty$, then Π is uniquely determined. This certainly is true if (2.1;1) with output W_0 is observable or if W_0 is simply non-singular. Using the time independent solution of (2.9) in the feedback law (2.8) gives the optimal control and trajectory as in the finite time horizon case.

3. FINITE DIMENSIONAL APPROXIMATIONS

Our goal is to construct systems of ordinary differential equations, such that their solutions approximate the solution to the hereditary control problem in Section 2. To this end, let Y^N , $N = 1, 2, \dots$ be a sequence of finite dimensional subspaces of $L^2(-h, 0; \mathbb{R}^n)$ with corresponding orthogonal projections p_1^N . Then $Z^N = \mathbb{R}^n \times Y^N$, $N = 1, 2, \dots$ are finite dimensional subspaces of M^2 with corresponding orthogonal projections $p^N \phi = (\phi^0, p_1^N \phi^1)$, $\phi \in M^2$. Suppose there is a sequence of linear operators $A^N: Z^N \rightarrow Z^N$ and let $S^N(t)$, $t \geq 0$ be the uniformly continuous semigroups on M^2 generated by the bounded linear operators $A^N p^N: M^2 \rightarrow Z^N$, i.e.,

$$S^N(t)\phi = e^{A^N p^N t} \phi, \quad \phi \in M^2.$$

Remark. We extend A^N to all of M^2 , because we want the semigroup $S^N(\cdot)$ acting on the whole space. Instead of letting the generator $A^N p^N = 0$ on $(Z^N)^\perp$, we equally well could choose another appropriate extension. All that is said about the control problems in Z^N and the corresponding semigroups $S^N(\cdot)$ in M^2 in this section, remains valid, if we take the generator of $S^N(\cdot)$ to be $A^N p^N - \alpha(I - p^N)$ with some $\alpha \in \mathbb{R}$. For simplicity of exposition, we shall make use of this possibility only at the end of the proof of Theorem 3.5 below.

We will use the following hypotheses in order to get the desired convergence of $S^N(t)$ to $S(t)$, the solution of semigroup generated by A as defined in Section 2.

(H1) There exists a constant $\omega \in \mathbb{R}$ such that

$$\langle A^N \phi, \phi \rangle_g \leq \omega \|\phi\|_g^2 \quad \text{for all } \phi \in Z^N, N = 1, 2, \dots .$$

(H2) There exists a subset $D \subseteq \text{dom } A$ and a real number $\lambda > \omega$, such that

(i) $(\lambda I - A)D$ is dense in M^2 ,

(ii) for all $\phi \in D$, $A^N p^N \phi \rightarrow A\phi$ as $N \rightarrow \infty$.

By the Lumer-Phillips theorem, (H1) implies

$$(3.1) \quad \|S^N(t)\phi\|_{M^2} \leq M e^{\omega t} \|\phi\|_{M^2}, \quad t \geq 0$$

for all $\phi \in M^2$ and some $M > 1$. Since $(\lambda I - A)^{-1}: M^2 \rightarrow \text{dom } A$ is continuous if $\lambda > \omega$, (H2) (i) implies that also D is dense in M^2 , so that we have the following version of the Trotter-Kato theorem [17, III. Th. 4.5].

Theorem 3.1: If the sequences $Z^N, A^N, N = 1, 2, \dots$ satisfy (H1) and (H2), then for all $\phi \in M^2$, $S^N(t)\phi \rightarrow S(t)\phi, N \rightarrow \infty$, uniformly in t on bounded intervals.

Observing $B\xi = (B_0\xi, 0) \in Z^N, \xi \in \mathbb{R}^m$ for all N , we take the input operators $B^N: \mathbb{R}^m \rightarrow Z^N$ as $B^N\xi = B\xi$ to define finite dimensional control systems on Z^N :

$$(\Sigma^N) \quad \dot{z}(t) = A^N z(t) + B^N u(t), \quad t \geq 0, \quad z(0) = p^N \phi,$$

where $u(\cdot) \in L^2_{loc}(0, \infty; \mathbb{R}^m)$ and $\phi \in M^2$. The solution to (Σ^N) is given by

$$z^N(t; p^N_\phi, u) = S^N(t)p^N_\phi + \int_0^t S^N(t-s)B^N u(s)ds.$$

As a consequence of Theorem 3.1, one derives (see for instance [14], Th. 4.2) convergence of $z^N(t; p^N_\phi, u)$ to $z(t; \phi, u)$, the solution of (Σ) given in (2.3):

Corollary 3.2. Assume (H1), (H2) and

$$(H3) \quad p^N_\phi \rightarrow \phi \text{ for all } \phi \in M^2.$$

Then for all $\phi \in M^2$

$$(3.2) \quad z^N(t; p^N_\phi, u) \rightarrow z(t; \phi, u), \text{ as } N \rightarrow \infty.$$

The limit is uniform in t on bounded intervals and u in bounded subsets of $L^2_{loc}(0, \infty; \mathbb{R}^m)$.

Writing $z^N(t; p^N_\phi, u) = (x^N(t), y^N(t)) \in Z^N$, with some $x^N(t) \in \mathbb{R}^n$, $y^N(t) \in Y^N$ (3.2) implies $x^N(t) \rightarrow x(t; \phi, u)$ uniformly in t on bounded intervals, where $x(t; \phi, u)$ is the solution of the hereditary system (2.1).

Seeking approximations for the optimal control problem (2.3), (2.6), we define the costs $J^N(u, \phi, T)$ by

$$(3.3) \quad J^N(u, \phi, T) = \langle z^N(T; p^N_\phi, u), Gz^N(T; p^N_\phi, u) \rangle + \int_0^T (\langle z^N(t; p^N_\phi, u), Wz^N(t; p^N_\phi, u) \rangle + u(t)^T Ru(t)) dt.$$

From the control theory of finite dimensional systems, it is well known that the optimal control minimizing (3.3) is given by

$$(3.4) \quad \bar{u}^N(t) = -R^{-1}(B^N)^* \Pi^N(t) z^N(t), \quad 0 \leq t \leq T$$

where $\bar{z}^N(t)$ is the solution of

$$\dot{z}^N(t) = (A^N - B^N R^{-1}(B^N)^* \Pi^N(t)) z^N(t), \quad z^N(0) = p^N_\phi,$$

and $\Pi^N(t): Z^N \rightarrow Z^N$ is the unique nonnegative selfadjoint solution of the Riccati differential equation on Z^N

$$(3.5) \quad \frac{d}{dt} \Pi^N(t) + (A^N)^* \Pi^N(t) + \Pi^N(t) A^N - \Pi^N(t) B^N R^{-1} (B^N)^* \Pi^N(t) + W^N = 0, \quad 0 \leq t \leq T,$$

$$\Pi^N(T) = G^N.$$

Here $G^N_\phi = G\phi$ and $W^N_\phi = W\phi$ for $\phi \in Z^N$. The optimal costs are given by $J^N(\bar{u}^N, \phi, T) = \langle p^N_\phi, \Pi^N(0) p^N_\phi \rangle$.

In applications, the original system (Σ) is controlled by use of the approximate feedback law (3.4) instead of (2.8), i.e., $\bar{u}(t)$ is replaced by

the so-called suboptimal control

$$(3.6) \quad \hat{u}^N(t) = -R^{-1}(B^N)^* \Pi^N(t) p^N \hat{z}^N(t), \quad 0 \leq t \leq T,$$

where $\hat{z}^N(t)$ is the mild solution of

$$\dot{z}(t) = (A - BR^{-1}(B^N)^* \Pi^N(t) p^N) z(t), \quad z(0) = \phi.$$

In order to establish convergence of the optimal and suboptimal controls, we have to deal with the M^2 -adjoint semigroups $S^N(\cdot)^*$ generated by the M^2 -adjoints $(A^N p^N)^*$ of $A^N p^N$. The adjoint of $A^N: Z^N \rightarrow Z^N$ is $(A^N)^*: Z^N \rightarrow Z^N$. Therefore, $(A^N p^N)^* = (p^N A^N p^N)^* = p^N (A^N)^* p^N = (A^N)^* p^N$. Since (H1) implies (3.1), the estimate (3.1) is valid also for $(A^N)^*$ if $A^N, N = 1, 2, \dots$ satisfy (H1). Thus, using Theorem 3.1 and Corollary 3.2, beside (H1), (H3) we need (H2) with A, A^N replaced by $A^*, (A^N)^*$ respectively (this will be denoted by (H2^{*})), to obtain $S^N(t)^* \phi \rightarrow S^*(t) \phi, \phi \in M^2$, uniformly in t on bounded intervals.

The following assertions were proved in [9] (also [14], Th. 4.3).

Theorem 3.3: Let (H1) - (H3) and (H2^{*}) hold. Then, as $N \rightarrow \infty$

$$(a) \quad \|\Pi^N(t) p^N - \Pi(t)\|_{L(M^2)} \rightarrow 0.$$

(b) For the controls and trajectories given in (3.4), (3.6), and (2.8)

$$\hat{u}^N(t) \rightarrow \bar{u}(t), \quad \bar{u}^N(t) \rightarrow \bar{u}(t), \quad \hat{z}^N(t) \rightarrow \bar{z}(t), \quad \bar{z}^N(t) \rightarrow \bar{z}(t)$$

the limits being uniform in $t, 0 \leq t \leq T$.

$$(c) \quad J^N(\hat{u}^N, \phi, T) \rightarrow J(\bar{u}, \phi, T), \quad J^N(\bar{u}^N, \phi, T) \rightarrow J(\bar{u}, \phi, T).$$

In the case of the infinite time horizon, we deal with the algebraic Riccati equations

$$(3.7) \quad (A^N)^* \Pi^N + \Pi^N A^N - \Pi^N B^N R^{-1} (B^N)^* \Pi^N + W^N = 0$$

on $Z^N, N = 1, 2, \dots$. If (A^N, B^N) is stabilizable, then there exists a nonnegative, selfadjoint solution Π^N of (3.7), governing the optimal feedback.

We will establish convergence of the Riccati operators Π^N again using Gibson's arguments [9]. However, this approach is based on the assumption that the stabilizability of the hereditary system implies that the systems (A^N, B^N) are stabilizable in a uniform sense with respect to N (see hypothesis (H5) below). As to the investigation of stabilizability or exponential stability of C_0 -semigroups there is a L^2 -stability criterion due to R. Datko [7]. We state here a special version (a proof may also be found in [19]).

Lemma 3.4: Let $S(t), t \geq 0$ be a C_0 -semigroup of bounded linear operators in a Banach space X satisfying

$$(3.8) \quad \|S(t)\|_{L(X)} \leq M_1 e^{\alpha_1 t}, \quad t \geq 0 \quad \text{and}$$

$$(3.9) \quad \int_0^\infty \|S(t)x\|_X^2 dt \leq c_1 \|x\|_X^2, \quad x \in X$$

for some constants $M_1, \alpha_1, c_1 > 0$. Then there exists an exponent
 $\alpha = \alpha(c_1, M_1, \alpha_1) > 0$ and a constant $M = M(c_1, M_1, \alpha_1) > 0$ such that

$$(3.10) \quad \|S(t)\|_{L(X)} \leq M e^{-\alpha t}, \quad t \geq 0.$$

Note that if we can prove (3.9) for the semigroups $S^N(\cdot)$ with c_1 independent on N , then, by (H1), (3.10) yields the exponential stability of $S^N(\cdot)$ uniformly with respect to N . Moreover, observe that the estimate (3.9) is equivalent to

$$(3.11) \quad \int_0^\infty |(S(t)x)^0|_{\mathbb{R}^n}^2 dt \leq c_2 \|x\|_{M^2}^2, \quad x \in M^2,$$

for some constant $c_2 > 0$, if $S(\cdot)$ is the solution semigroup on M^2 as defined in Section 2. By Fubini's theorem, this equivalence follows directly from the state concept (2.2) and is a special feature of the semigroup associated with the retarded functional differential equation (2.1;1).

As far as we want the semigroups $S^N(\cdot)$ to be suitable approximations for $S(\cdot)$, it seems to be natural to demand the equivalence of (3.9) and (3.11) also with regard to $S^N(\cdot)$. We call this the VDP-property of the approximations (meaning the vector-component dominance is preserved). It plays an important role for our approach to stability questions in context with the infinite time horizon control problem.

Suppose for N sufficiently large there exists a solution Π^N to the N th algebraic Riccati equation (3.7). Then with

$$\bar{A}^N = A^N - B^N R^{-1} (B^N)^* \Pi^N$$

$\bar{A}^N p^N$ generates an exponentially stable semigroup $\bar{S}^N(\cdot)$ on M^2 . We introduce the projection $V: M^2 \rightarrow \mathbb{R}^n$

$$V(\phi^0, \phi^1) = \phi^0$$

and want the following hypothesis to be valid:

(H4) Provided N is sufficiently large, there exists a $c_1 \geq 0$, independent on N , such that for all $\phi \in Z^N$

$$\int_0^\infty \|\bar{S}^N(t)\phi\|_{M^2}^2 dt \leq c_1 \|\phi\|_{M^2}^2$$

if and only if there exists a $c_2 \geq 0$, independent on N , such that for all $\phi \in Z^N$

$$\int_0^\infty |V\bar{S}^N(t)\phi|_{\mathbb{R}^n}^2 dt \leq c_2 \|\phi\|_{M^2}^2.$$

If Π is a nonnegative selfadjoint solution to the algebraic Riccati equation (2.9) for the hereditary control problem, define the operators

$\tilde{A}^N: Z^N \rightarrow Z^N$, $N = 1, 2, \dots$ by

$$\tilde{A}^N = A^N - BR^{-1}B^*\Pi$$

and let $\tilde{S}^N(\cdot)$ be the uniformly continuous semigroup on M^2 generated by $\tilde{A}^N p^N$, i.e., $\tilde{S}^N(t) = e^{\tilde{A}^N p^N t}$, $t \geq 0$.

Intending to provide the existence and uniform boundedness of the operators Π^N , we demand:

(H5) If the hereditary system (Σ) is stabilizable, then there exist constants $M, \beta > 0$ such that for N sufficiently large

$$\|\tilde{S}^N(t)\|_{Z^N} \leq M e^{-\beta t}, \quad t \geq 0.$$

Theorem 3.5: Let (H1) - (H5), (H2*) hold and assume W_0 is nonsingular. If the hereditary system is stabilizable, then

(a) for N sufficiently large there exists a solution Π^N to the N th algebraic Riccati equation (3.7) and

$$\|\Pi^N - \Pi\|_{L(M^2)} \rightarrow 0, \quad N \rightarrow \infty.$$

(b) The optimal and suboptimal controls and trajectories and the corresponding costs converge as in Theorem 3.3 (b), (c).

Proof: As in [9, Th. 7.4], we first consider the N th problem (Σ^N) with initial value $\phi \in M^2$, when it is controlled by the feedback $\tilde{u}^N(t) = -R^{-1} B^* \Pi^N z(t)$. The evolution of the state in time is then described by $\tilde{z}^N(t) = \tilde{S}^N(t) p^N \phi$ and the corresponding costs can be estimated using (H5):

$$J^N(\tilde{u}^N, \phi) = \int_0^\infty (\langle \tilde{z}^N(t), W \tilde{z}^N(t) \rangle + \tilde{u}^N(t)^T R \tilde{u}^N(t)) dt$$

$$\leq \frac{M}{2\beta} (|W_0| + |R^{-1}| |B_0|^2 \|\Pi\|^2) \|\phi\|^2.$$

Thus, there exists a nonnegative selfadjoint solution Π^N of the Nth algebraic Riccati equation for N sufficiently large and

$$\langle P^N \phi, \Pi^N P^N \phi \rangle \leq J^N(\tilde{u}^N, \phi) \leq c_1 \|\phi\|^2$$

with some constant c_1 , which does not depend on N . Therefore, there is an index N_0 such that

$$(3.12) \quad \|\Pi^N P^N\| \leq c_1, \quad N \geq N_0.$$

The convergence statement in (a) now follows from [9, Th. 6.9], once that we have shown

$$(3.13) \quad \|\bar{S}^N(t)\| \leq \bar{M} e^{-\alpha t}, \quad t \geq 0, \quad N \geq N_0$$

for some constants $\bar{M}, \alpha > 0$.

Since $W_0 > 0$ we have $|\xi|^2 \leq \mu^{-1} \xi^T W_0 \xi$, $\xi \in \mathbb{R}^n$, where μ is the minimum eigenvalue of W_0 . Following the arguments given in [9] (proof of Th. 7.5) let $\phi \in Z^N$ and define $\bar{z}^N(t) = \bar{S}^N(t)\phi = (x^N(t), y^N(t))$ with $x^N(t) \in \mathbb{R}^n$, $y^N(t) \in Y^N$, $t \geq 0$. Further note that $\langle R^{-1} B^* \Pi^N \bar{z}^N(t), B^* \Pi^N \bar{z}^N(t) \rangle \geq 0$, \mathbb{R}^n so that

$$\begin{aligned} \int_0^\infty |x^N(t)|^2 dt &\leq \mu^{-1} \int_0^\infty x^N(t)^T W_0 x^N(t) dt \\ &\leq \mu^{-1} \int_0^\infty \langle \bar{z}^N(t), (W + \Pi^N B R^{-1} B^* \Pi^N) \bar{z}^N(t) \rangle dt \end{aligned}$$

$$= \mu^{-1} \langle \phi, \int_0^{\infty} (\bar{S}^N)^*(t)(W + \Pi^N BR^{-1} B^* \Pi^N) \bar{S}^N(t) \phi dt \rangle = \mu^{-1} \langle \phi, \Pi^N \phi \rangle,$$

applying [9], Cor. 4.2 to the semigroups $S^N(\cdot)$. It follows $\int_0^{\infty} |V \bar{S}^N(t) \phi|^2 dt \leq \mu^{-1} c_1 \|\phi\|^2$, $N \geq N_0$ and, by hypothesis (H4), $\int_0^{\infty} \|\bar{S}^N(t) \phi\|^2 dt \leq c_2 \|\phi\|^2$ with some $c_2 > 0$. Also, because \bar{S}^N is generated by $(A^N - BR^{-1} B^* \Pi^N) p^N$, (3.1) and (3.12) imply (e.g. [17], III. Th. 1.1) the existence of constants M_1, α_1 such that $\|\bar{S}^N(t)\| \leq M_1 e^{\alpha_1 t}$, $t \geq 0, N \geq N_0$. Now Lemma 3.4 assures that there are constants $M, \alpha > 0$ such that

$$\|\bar{S}^N(t) \Big|_{Z^N}\| \leq M e^{-\alpha t}, \quad t \geq 0, \quad N \geq N_0.$$

But this proves (3.13), since, as we mentioned in the remark above, we may replace $A^N p^N$ by $A^N p^N - \alpha(I - p^N)$, so that $\|\bar{S}^N(t) \Big|_{(Z^N)^\perp}\| \leq e^{-\alpha t}$, while the finite dimensional control problems on Z^N remain completely unchanged. Statement (b) follows from (a) in a manner similar to that in the finite time horizon case (see [9] or [14]).

4. THE PIECEWISE LINEAR APPROXIMATION SCHEME

While the foregoing statements are of rather general nature, this section presents a special approximation scheme using so-called piecewise linear functions. We prove via several lemmas that this scheme satisfies the hypotheses (H1) - (H3) and (H2*). Then we show that it has the VDP-property, so that (H4) is valid. Furthermore, we investigate the characteristic matrix of the approximate systems in order to establish results on the eigenvalue behavior

when the approximation index increases. This enables us to conclude (H5). Finally, we present some of our numerical findings.

4.1 Projection onto spaces of piecewise linear functions

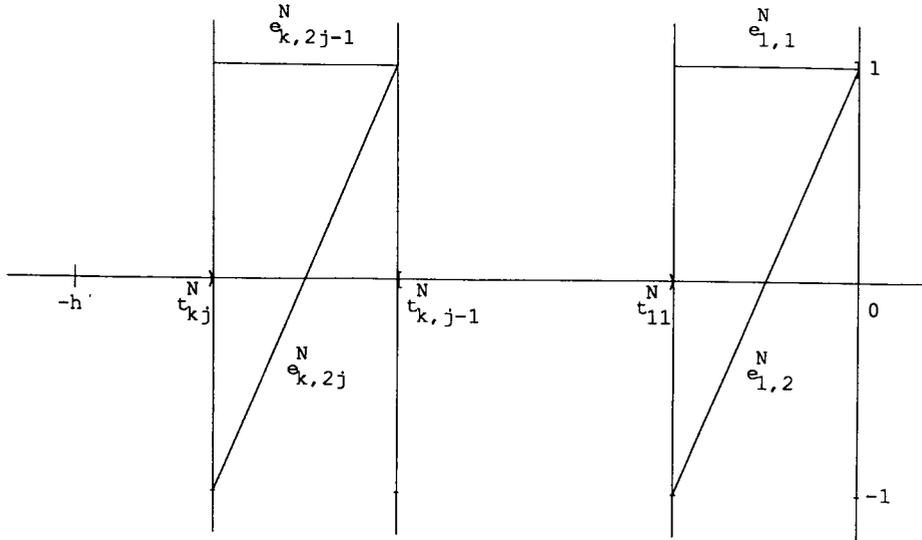
Corresponding to the discrete delays h_k in (2.1;1), for $N = 1, 2, \dots$ we subdivide the intervals $I_k = [-h_k, -h_{k-1})$, $k = 2, \dots, p$ and $I_1 = [-h_1, 0]$ into the subintervals $I_{kj}^N = [t_{kj}^N, t_{k,j-1}^N)$, $k = 1, \dots, p$, $j = 1, \dots, N$ (with the exception $I_{11}^N = [t_{11}^N, 0]$) by defining the meshpoints

$$t_{kj}^N = -h_{k-1} - jr_k/N, \quad j = 0, \dots, N,$$

where $r_k = h_k - h_{k-1}$, $k = 1, \dots, p$. For each $N \in \mathbb{N}$ the set Y^N of all functions $[-h, 0] \rightarrow \mathbb{R}^1$ that are polynomials of degree one on every interval I_{kj}^N is commonly called a space of piecewise linear functions on $[-h, 0]$. A basis of Y^N is given, written in a simplified notation, by the $2pN$ matrix functions

$$(4.1) \quad \begin{aligned} e_{k,2j-1}^N(s) &= \chi_{I_{kj}^N}^N(s) \cdot I, \\ e_{k,2j}^N(s) &= \left(\frac{2N}{r_k}(s+h_{k-1})+2j-1\right) \chi_{I_{kj}^N}^N(s) \cdot I, \end{aligned} \quad k = 1, \dots, p, \quad j = 1, \dots, N$$

where I denotes the $n \times n$ identity matrix. The following diagram illustrates the definition of the basis elements.



The pairs $\hat{e}_0 = (I, 0)$ and $\hat{e}_{kj}^N = (0, e_{kj}^N)$ are an orthogonal basis of the $n(2pN+1)$ dimensional product space $Z^N = \mathbb{R}^n \times Y^N$, $N = 1, 2, \dots$. The orthogonal projections $p^N: M^2 \rightarrow Z^N$ are of the form $p^N(\phi^0, \phi^1) = (\phi^0, p_1^N \phi^1)$, where p_1^N is the L^2 -orthogonal projection from L^2 onto Y^N . With

$$(4.2) \quad Q^N = (\langle \hat{e}_{kj}^N, \hat{e}_{m\ell}^N \rangle) =$$

$$= \text{diag}(1, \frac{r_1}{N}, \frac{r_1}{3N}, \dots, \frac{r_1}{N}, \frac{r_1}{3N}, \frac{r_2}{N}, \frac{r_2}{3N}, \dots, \frac{r_p}{N}, \frac{r_p}{3N}) \otimes I$$

we have

$$(4.3) \quad \langle \phi, \psi \rangle = \alpha^N(\phi)^T Q^N \alpha^N(\psi), \quad \phi, \psi \in Z^N,$$

$$(4.4) \quad \alpha^N(p^N \phi) = (Q^N)^{-1} \text{col}(\phi^0, \langle e_{11}^N, \phi^1 \rangle_2, \dots, \langle e_{p,2N}^N, \phi^1 \rangle_2), \quad \phi \in M^2$$

where the components of the coefficient vector $\alpha^N(p^N \phi)$ are given by

$\alpha_0 = \phi^0$ and

$$(4.5) \quad \alpha_{k,2j-1}^N(p^N \phi) = \frac{N}{r_k} \int_{I_{kj}^N} \phi^1(s) ds, \quad k = 1, \dots, p, \quad j = 1, \dots, N.$$

$$\alpha_{k,2j}^N(p^N \phi) = \frac{3N}{r_k} \int_{I_{kj}^N} e_{k,2j}^N(s) \phi^1(s) ds.$$

We frequently will use the abbreviations $\phi_{kj}^N = \alpha_{kj}^N(p^N \phi)$ and $\phi^N = p_1^N \phi^1$.

Note that

$$(4.6) \quad |\phi_{k,2j-1}^N| \leq \left(\frac{N}{r_k}\right)^{1/2} \|\phi^1\|_2$$

$$|\phi_{k,2j}^N| \leq \left(\frac{3N}{r_k}\right)^{1/2} \|\phi^1\|_2.$$

$k = 1, \dots, p, \quad j = 1, \dots, N$

Obviously, the spaces Z^N are not contained in the domain of the generator A of the hereditary semigroup, since the elements of Y^N are not differentiable on $[-h, 0]$. Nevertheless, the action of A and A^* can be approximated by operators on Z^N imitating, heuristically speaking, the delta distributions in the derivatives of discontinuous functions by operators δ^N in Y^N . Following the ideas developed in [14], we need the operators δ^N for each point, where the piecewise linear functions in Y^N may have jumps.

We define $\delta_{kj}^{N-}, \delta_{kj}^{N+}: \mathbb{R}^n \rightarrow Y^N, N = 1, 2, \dots$ by

$$\delta_{kj}^{N-}(\xi) = \left(\frac{N}{r_k} e_{k,2j+1}^N + \frac{3N}{r_k} e_{k,2j+2}^N\right)\xi, \quad k = 1, \quad j = 0, \dots, N-1$$

$$k = 2, \dots, p, \quad j = 1, \dots, N-1$$

$$\delta_{kN}^{N-}(\xi) = \left(\frac{N}{r_{k+1}} e_{k+1,1}^N + \frac{3N}{r_{k+1}} e_{k+1,2}^N\right)\xi, \quad k = 1, \dots, p-1$$

and

$$\delta_{kj}^{N+}(\xi) = \left(\frac{N}{r_k} e_{k,2j-1}^N - \frac{3N}{r_k} e_{k,2j}^N \right) \xi, \quad k = 1, \dots, p, \quad j = 1, \dots, N.$$

Proposition 4.1. (a) For any $\xi \in \mathbb{R}^n$ and $\phi^1 \in L^2(-h, 0; \mathbb{R}^n)$

$$(4.7) \quad \langle \delta_{kj}^{N-}(\xi), \phi^1 \rangle_2 = \xi^T \phi^N(t_{kj}^{N-}), \quad \begin{array}{l} k = 1, j = 0, \dots, N \\ k = 2, \dots, p-1, j = 1, \dots, N \\ k = p, j = 1, \dots, N-1 \end{array}$$

$$(4.8) \quad \langle \delta_{kj}^{N+}(\xi), \phi^1 \rangle_2 = \xi^T \phi^N(t_{kj}^N), \quad k = 1, \dots, p, \quad j = 1, \dots, N.$$

In (4.7) and throughout the paper, we use the notation $\phi^N(t_{kj}^{N-})$ for the left side limit of ϕ^N at t_{kj}^N .

(b) The norms of the δ -operators are

$$\|\delta_{kj}^{N-}\| = 2\left(\frac{N}{r_k}\right)^{1/2}, \quad \begin{array}{l} k = 1, j = 0, \dots, N-1 \\ k = 2, \dots, p, j = 1, \dots, N-1 \end{array}$$

$$\|\delta_{kN}^{N-}\| = 2\left(\frac{N}{r_{k+1}}\right)^{1/2}, \quad k = 1, \dots, p-1$$

$$\|\delta_{kj}^{N+}\| = 2\left(\frac{N}{r_k}\right)^{1/2}, \quad k = 1, \dots, p, \quad j = 1, \dots, N.$$

Proof: (a) Let $1 \leq k \leq p$ and $1 \leq j \leq N-1$. Using (4.5) and observing that

$$e_{k,2j-1}^N(t_{k,j-1}^{N-}) = e_{k,2j}^N(t_{k,j-1}^{N-}) = I,$$

we get

$$\begin{aligned} \langle \delta_{kj}^{N-}(\xi), \phi^1 \rangle_2 &= \int_{-h}^0 \xi^T \left(\frac{N}{r_k} e_{k,2j+1}^N + \frac{3N}{r_k} e_{k,2j+2}^N \right)(s) \phi^1(s) ds \\ &= \xi^T (\phi_{k,2j+1}^N + \phi_{k,2j+2}^N) = \xi^T \phi^N(t_{kj}^{N-}). \end{aligned}$$

The proof for $k = 1, \dots, p-1, j = N$ and for δ_{10}^{N-} is analogous. The statement on δ_{kj}^{N+} follows similarly from the fact that $e_{k,2j-1}^N(t_{kj}^N) = I$ and $e_{k,2j}^N(t_{kj}^N) = -I, k = 1, \dots, p, j = 1, \dots, N.$

(b) Since the basis of Y^N is an orthogonal set, we have

$$\begin{aligned} \|\delta_{kj}^{N+}(\xi)\|_2^2 &= |\xi|^2 \left(\left(\frac{N}{r_k}\right)^2 \|e_{k,2j-1}^N\|^2 + \left(\frac{3N}{r_k}\right)^2 \|e_{k,2j}^N\|^2 \right) \\ &= |\xi|^2 \left(\frac{N}{r_k} + \frac{3N}{r_k} \right) \quad \text{by (4.2)}. \end{aligned}$$

The proof for δ_{kj}^{N-} is analogous. ■

Next we establish convergence results for the piecewise linear projections of sufficiently smooth functions.

Lemma 4.2: For $\phi^1 \in \{\psi^1 \in L^2(-h, 0; \mathbb{R}^n) \mid \psi^1|_{I_k} \in W^{2,\infty}(I_k; \mathbb{R}^n),$
 $k = 1, \dots, p\}$

$$(a) \quad \|\phi^N - \phi^1\|_\infty \leq \frac{c_1}{N^2} \|\ddot{\phi}^1\|_\infty$$

$N = 1, 2, \dots$

$$(b) \quad \|D^+ \phi^N - D\phi^1\|_\infty \leq \frac{c_2}{N} \|\ddot{\phi}^1\|_\infty .$$

The constants c_1, c_2 do not depend on N or ϕ^1 .

Proof: The proof is based on an application of the Peano Kernel Theorem (see for instance [20], Th. 1.3). Let $s \in I_{kj}^N$ and define the linear functional $F_s: W^{2,\infty}(I_{kj}^N; \mathbb{R}^n) \rightarrow \mathbb{R}^n$ by

$$F_s(\phi^1) = \phi^1(s) - \phi^N(s).$$

$F_s(p) = 0$ for all polynomials p of degree one, so the said theorem assures

$$(4.9) \quad \begin{aligned} |\phi^1(s) - \phi^N(s)| &= |F_s(\phi^1)| = \left| F_s \left(\int_{I_{kj}^N} \ddot{\phi}^1(\tau) (\cdot - \tau)_+ d\tau \right) \right| \\ &\leq \left| \int_{I_{kj}^N} \ddot{\phi}^1(\tau) (s - \tau)_+ d\tau \right| + \left| \left(\int_{I_{kj}^N} \ddot{\phi}^1(\tau) (\cdot - \tau)_+ d\tau \right)^N(s) \right| \end{aligned}$$

where

$$(s - \tau)_+ = \begin{cases} s - \tau, & \tau \leq s \\ 0 & , \tau > s \end{cases}.$$

Now

$$(4.10) \quad \left| \int_{I_{kj}^N} \ddot{\phi}^1(\tau) (s - \tau)_+ d\tau \right| = \left| \int_{t_{kj}^N}^s \ddot{\phi}^1(\tau) (s - \tau) d\tau \right| \leq \frac{1}{2} \left(\frac{\bar{r}}{N} \right)^2 \|\ddot{\phi}^1\|_\infty,$$

where $\bar{r} = \max\{r_k, k = 1, \dots, p\}$. To estimate the second term in (4.9), we define

$$(4.11) \quad \psi^1(t) = \int_{I_{kj}^N} \ddot{\phi}^1(\tau) (t - \tau)_+ d\tau, \quad t \in I_{kj}^N, \quad \psi^1(t) = 0 \quad \text{elsewhere,}$$

so that

$$\begin{aligned} & \left| \left(\int_{I_{kj}} \ddot{\phi}^1(\tau)(\cdot - \tau)_+ d\tau \right)^N(s) \right| = |\psi^N(s)| \\ & = \left| \sum_{m=1}^p \sum_{\ell=1}^{2N} e_{m\ell}^N(s) \psi_{m\ell}^N \right| \leq |\psi_{k,2j-1}^N| + |\psi_{k,2j}^N|. \end{aligned}$$

But from (4.11)

$$\|\psi^1\|_2 \leq \left(\frac{1}{20}\right)^{1/2} \left(\frac{r_k}{N}\right)^{5/2} \|\ddot{\phi}^1\|_\infty.$$

Hence, by (4.6), statement (a) follows. Similar methods for the functional

$$\begin{aligned} G_s: W^{2,\infty}(I_{kj}^N; \mathbb{R}^n) \rightarrow \mathbb{R}^n, \quad G_s(\phi^1) = D\phi^1(s) - D^+\phi^N(s), \quad \text{and the fact that} \\ D^+\psi^N = \frac{2N}{r_k} e_{k,2j-1}^N \psi_{k,2j}^N \quad \text{yield (b).} \end{aligned}$$

As an immediate consequence from Lemma 4.2 (a), we see that the subspaces Z^N defined in this section satisfy hypothesis (H3), since the set $\{(\phi^0, \phi^1) \in M^2 \mid \phi^0 \in \mathbb{R}^n, \phi^1|_{I_k} \in W^{2,\infty}(I_k; \mathbb{R}^n), k = 1, \dots, p\}$ is dense in M^2 and $\|p^N\| = 1$ for all N .

4.2 The approximating semigroups and their generators

Definition 4.3. For $\phi = (\phi^0, \phi^1) \in Z^N$ we define

$$\begin{aligned} A^N(\phi^0, \phi^1) = & (A_0\phi^0 + \sum_{k=1}^p A_k\phi^1(-h_k) + \int_{-h}^0 A_{01}(s)\phi^1(s)ds, D^+\phi^1 + \delta_{10}^{N-}(\phi^0 - \phi^1(0))) \\ & + \sum_{k=1}^{p-1} \sum_{j=1}^N \delta_{kj}^{N-}(\phi^1(t_{kj}^N) - \phi^1(t_{kj}^{N-})) + \sum_{j=1}^{N-1} \delta_{pj}^{N-}(\phi^1(t_{pj}^N) - \phi^1(t_{pj}^{N-})). \end{aligned}$$

Since $D^+\phi^1 \in Y^N$ for $\phi^1 \in Y^N$ it is clear that A^N is a linear operator $Z^N \rightarrow Z^N$.

Lemma 4.4: The adjoint of A^N is given by

$$(A^N)^*(\psi^0, \psi^1) = (A_{01}^T \psi^0 + \psi^1(0), (A_{01}^T \psi^0)^N - D^+ \psi^1 + \sum_{k=1}^p \delta_{kN}^{N+}(A_k^T \psi^0) + \sum_{k=1}^{p-1} \sum_{j=1}^N \delta_{kj}^{N+} \psi^1(t_{kj}^{N-}) - (\psi^1(t_{kj}^N)) + \sum_{j=1}^{N-1} \delta_{pj}^{N+} (\psi^1(t_{pj}^{N-}) - \psi^1(t_{pj}^N)) - \delta_{pN}^{N+} (\psi^1(-h)))$$

for $(\psi^0, \psi^1) \in Z^N$.

Proof: Evaluating $\langle (\psi^0, \psi^1), A^N(\phi^0, \phi^1) \rangle$ for $\psi, \phi \in Z^N$ we get by Proposition 4.1.

$$\psi^{0T} \sum_{k=1}^p A_k \phi^1(-h_k) = \sum_{k=1}^p \langle \delta_{kN}^{N+}(A_k^T \psi^0), \phi^1 \rangle_2$$

and, since $\phi^1 \in Y^N$,

$$\int_{-h}^0 \psi^{0T} A_{01}(s) \phi^1(s) ds = \langle A_{01}^T \psi^0, \phi^1 \rangle_2 = \langle (A_{01}^T \psi^0)^N, \phi^1 \rangle_2.$$

Integrating by parts the term involving $D^+ \phi^1$ yields

$$\langle \psi^1, D^+ \phi^1 \rangle_2 = \sum_{k=1}^p \sum_{j=0}^{N-1} \psi^1(t_{kj}^{N-})^T \phi^1(s) - \sum_{k=1}^p \sum_{j=1}^N \psi^1(t_{kj}^N)^T \phi^1(t_{kj}^N) - \langle D^+ \psi^1, \phi^1 \rangle_2.$$

Furthermore,

$$\langle \psi^1, \delta_{10}^{N-}(\phi^0 - \phi^1(0)) \rangle_2 = (\phi^0 - \phi^1(0))^T \psi^1(0^-) = \psi^1(0)^T \phi^0 - \psi^1(0)^T \phi^1(0),$$

so that

$$\begin{aligned} \langle (A^N)^*(\psi^0, \psi^1), (\phi^0, \phi^1) \rangle_{M^2} &= (A_0^T \psi^0 + \psi^1(0))^T \phi^0 \\ &+ \left\langle \sum_{k=1}^p \delta_{kN}^{N+} (A_k^T \psi^0) + (A_{01}^T \psi^0)^N - D^+ \psi^1, \phi^1 \right\rangle_2 + \Delta, \end{aligned}$$

where Δ is given by

$$\begin{aligned} \Delta &= -\psi^1(0)^T \phi^1(0) + \sum_{k=1}^p \sum_{j=1}^{N-1} \psi^1(t_{kj}^{N-})^T \phi^1(s) - \sum_{k=1}^p \sum_{j=1}^N \psi^1(t_{kj}^N)^T \phi^1(t_{kj}^N) \\ &+ \sum_{k=1}^{p-1} \sum_{j=1}^N \langle \psi^1, \delta_{kj}^{N-} (\phi^1(t_{kj}^N) - \phi^1(t_{kj}^{N-})) \rangle_2 + \sum_{j=1}^{N-1} \langle \psi^1, \delta_{pj}^{N-} (\phi^1(t_{pj}^N) - \phi^1(t_{pj}^{N-})) \rangle_2. \end{aligned}$$

The last two sums are transformed using (4.7). Observing $\phi^1(0^-) = \phi^1(0)$ and $t_{kN}^N = t_{k+1,0}^N$, $k = 1, \dots, p$, then arranging Δ appropriately we get

$$\begin{aligned} \Delta &= \sum_{k=1}^{p-1} \sum_{j=1}^N (\psi^1(t_{kj}^{N-}) - \psi^1(t_{kj}^N))^T \phi^1(t_{kj}^N) \\ &+ \sum_{j=1}^{N-1} (\psi^1(t_{pj}^{N-}) - \psi^1(t_{pj}^N))^T \phi^1(t_{pj}^N) - \psi^1(-h)^T \phi^1(-h) \end{aligned}$$

and the result follows by applications of (4.8). ■

The next lemma shows that the operators A^N satisfy the uniform dissipativity condition (H1) in the space M_g^2 defined in Section 2.

Lemma 4.5: For all N and all $\phi \in Z^N$, $\langle A^N \phi, \phi \rangle_g \leq \omega \|\phi\|_g^2$, with
 ω given in (2.4).

Proof: As in Proposition 4.1, it is easy to verify that, with respect to the weighting function $g(s) = p - k + 1$, $s \in I_k$, $k = 1, \dots, p$, we have

$$\langle \delta_{10}^{N-}(\xi), \phi^1 \rangle_{L_g^2} = p \cdot \xi^T \phi^N(0^-),$$

$$\langle \delta_{kj}^{N-}(\xi), \phi^1 \rangle_{L_g^2} = (p-k+1) \xi^T \phi^N(t_{kj}^{N-}), \quad k = 1, \dots, p, j=1, \dots, N-1,$$

$$\langle \delta_{kN}^{N-}(\xi), \phi^1 \rangle_{L_g^2} = (p-k) \xi^T \phi^N(-h_k^-), \quad k = 1, \dots, p-1$$

for all $\phi^1 \in L^2(-h, 0; \mathbb{R}^1)$ and $\xi \in \mathbb{R}^1$. From these equations and the definition of A^N , it follows

$$\begin{aligned} \langle A^N \phi, \phi \rangle_g &\leq (A_0 \phi^0)^T \phi^0 + \sum_{k=1}^p |A_k| |\phi^1(-h_k)| |\phi^0| \\ &+ \left| \int_{-h}^0 A_{01}(s) \phi^1(s) ds \right| |\phi^0| + \sum_{k=1}^p (p-k+1) \sum_{j=1}^N \int_{I_{kj}} (D^+ \phi^1(s))^T \phi^1(s) ds \\ &+ p(\phi^0 - \phi^1(0))^T \phi^1(0^-) + \sum_{k=1}^p \sum_{j=1}^{N-1} (p-k+1) (\phi^1(t_{kj}^N) - \phi^1(t_{kj}^{N-}))^T \phi^1(t_{kj}^{N-}) \\ &+ \sum_{k=1}^{p-1} (p-k) (\phi^1(t_{kN}^N) - \phi^1(t_{kN}^{N-}))^T \phi^1(t_{kN}^{N-}) \end{aligned}$$

for $\phi \in Z^N$. Using

$$\int_{I_{kj}} (D^+ \phi^1(s))^T \phi^1(s) ds = \frac{1}{2} (|\phi^1(t_{k,j-1}^{N-})|^2 - |\phi^1(t_{kj}^N)|^2)$$

and the inequality $\xi^T \eta \leq \frac{1}{2} (|\xi|^2 + |\eta|^2)$, $\xi, \eta \in \mathbb{R}^1$, we get

$$\begin{aligned}
 \langle A^N \phi, \phi \rangle_g &\leq (A_0 \phi^0)^T \phi^0 + \frac{1}{2} \sum_{k=1}^p (|A_k|^2 \|\phi\|_{M^2}^2 + |\phi^1(-h_k)|^2) \\
 &+ \|A_{01}\|_2 \|\phi^1\|_2 \|\phi\|_{M^2} + \frac{1}{2} \sum_{k=1}^p (p-k+1) \sum_{j=0}^{N-1} |\phi^1(t_{kj}^N)|^2 \\
 &- \frac{1}{2} \sum_{k=1}^p (p-k+1) \sum_{j=1}^N |\phi^1(t_{kj}^N)|^2 + p(|\phi^0| |\phi^1(0)| - |\phi^1(0)|^2) \\
 &+ \frac{1}{2} \sum_{k=1}^p (p-k+1) \sum_{j=1}^{N-1} |\phi^1(t_{kj}^N)|^2 - \frac{1}{2} \sum_{k=1}^p (p-k+1) \sum_{j=1}^{N-1} |\phi^1(t_{kj}^N)|^2 \\
 &+ \frac{1}{2} \sum_{k=1}^{p-1} (p-k) |\phi^1(t_{kN}^N)|^2 - \frac{1}{2} \sum_{k=1}^{p-1} (p-k) |\phi^1(t_{kN}^N)|^2 \\
 &\leq (A_0 \phi^0)^T \phi^0 + \frac{p}{2} |\phi^0|^2 + \left(\frac{1}{2} \sum_{k=1}^p |A_k|^2 + \|A_{01}\|_{L^2} \right) \|\phi\|_g^2 \\
 &+ \frac{1}{2} \sum_{k=1}^p |\phi^1(-h_k)|^2 - \frac{p}{2} |\phi^1(0)|^2 - \frac{1}{2} \sum_{k=1}^p (p-k+1) |\phi^1(t_{kN}^N)|^2 \\
 &+ \frac{1}{2} \sum_{k=1}^p (p-k+1) |\phi^1(t_{k0}^N)|^2 + \frac{1}{2} \sum_{k=1}^{p-1} (p-k) |\phi^1(t_{kN}^N)|^2 \\
 &- \frac{1}{2} \sum_{k=1}^{p-1} (p-k) |\phi^1(t_{kN}^N)|^2 \leq \omega \|\phi\|_g,
 \end{aligned}$$

because, due to the weights, all terms except the first three neutralize each other. ■

Looking for appropriate sets to be used in (H2) and (H2*), we define

$$D = \{(\phi^1(0), \phi^1) \in M^2 \mid \phi^1 \in W^{2,\infty}(-h, 0; \mathbb{R}^n)\}$$

and

$$D^* = \{(\psi^0, \psi^1) \in \text{dom } A^* \mid \psi^1 \in W^{2,\infty}(I_k; \mathbb{R}^n), k = 1, \dots, p\}.$$

The following Lemma occupies (H2) (ii) and (H2*) (ii).

Lemma 4.6: (a) There is a constant $c > 0$, such that for all N and
all $\phi \in D$

$$\|A_p^N \phi - A\phi\| \leq \frac{c}{N} \|\phi\|_\infty.$$

(b) For all $\psi \in D^*$

$$\|(A^N)^* p^N \psi - A^* \psi\| \rightarrow 0, \text{ as } N \rightarrow \infty.$$

Proof: (a) From the definition of A and A^N we have the estimate

$$\begin{aligned} \|A_p^N \phi - A\phi\| &\leq \sum_{k=1}^p |A_k| |\phi^N(-h_k) - \phi^1(-h_k)| \\ &+ \int_{-h}^0 |A_{01}(s)(\phi^N(s) - \phi^1(s))| ds + \|D^+ \phi^N - D\phi^1\|_2 \\ &+ \|\delta_{10}^{N-} (\phi^0 - \phi^N(0))\|_2 + \|\Delta^N(\phi^1)\|_2 \end{aligned}$$

where

$$\begin{aligned} \Delta^N(\phi^1) &= \sum_{k=1}^p \sum_{j=1}^{N-1} \delta_{kj}^{N-} (\phi^N(t_{kj}^N) - \phi^N(t_{kj}^{N-})) + \sum_{k=1}^{p-1} \delta_{kN}^{N-} (\phi^N(t_{kN}^N) - \phi^N(t_{kN}^{N-})) \\ &= \Delta_1^N(\phi^1) + \Delta_2^N(\phi^1). \end{aligned}$$

By proposition 4.1 and Lemma 4.2 for $\phi \in D$ this yields

$$\|A_p^N \phi - A\phi\| \leq \left(\frac{c_1}{N^2} + \frac{c_2}{N} + 2 \left(\frac{N}{T_1} \right)^{1/2} \frac{c_3}{N^2} \right) \|\phi\|_\infty + \|\Delta^N(\phi^1)\|_2$$

with some constants c_1, c_2, c_3 . Since ϕ^1 is continuous on $[-h, 0]$ one has

$$|\phi^N(t_{kN}^N) - \phi^N(t_{kN}^{N-})| \leq 2\|\phi^N - \phi^1\|_\infty \leq \frac{C_4}{N^2} \|\ddot{\phi}^1\|_\infty, \quad k = 1, \dots, p-1$$

so that

$$\|\Delta_2^N(\phi^1)\|_2 \leq 2\left(\frac{N}{\underline{r}}\right)^{1/2} (p-1) \frac{\text{const}}{N^2} \|\ddot{\phi}^1\|_\infty,$$

with $\underline{r} = \min \{r_k, k = 1, \dots, p\}$. Taking advantage of the orthogonality of $e_{11}^N, \dots, e_{p, 2N}^N$ in L^2 , we have for $\xi \in \mathbb{R}^n$

$$\left\| \sum_{k=1}^p \sum_{j=1}^{N-1} \delta_{kj}^{N-}(\xi) \right\|_2^2 \leq 4p(N-1) \frac{N}{\underline{r}} |\xi|^2.$$

Hence,

$$\|\Delta_1^N(\phi^1)\|_2 \leq 2(p(N-1))^{1/2} \frac{N}{\underline{r}} \max_{\substack{k=1, \dots, p \\ j=1, \dots, N}} |\phi^N(t_{kj}^N) - \phi^N(t_{kj}^{N-})| \leq \frac{\text{const}}{N} \|\ddot{\phi}^1\|_\infty.$$

This proves (a).

(b) From Lemma 4.4 and the definition of A^* we have

$$\begin{aligned} \|(A^N)^* \psi^N - A^* \psi\| &\leq |A_0^T \psi^0 + \psi^N(0) - (A_0^T \psi^0 + \psi^1(0))| \\ &+ \|(A_{01}^T \psi^0)^N - A_{01}^T \psi^0\|_2 + \|D^+ \psi^N - D(\psi^1 + \sum_{k=1}^{p-1} A_k^T \psi^0 \chi_{[-h, -h_k]})\|_2 + \|\Delta^{N*}(\psi)\|_2, \end{aligned}$$

where

$$\Delta^{N*}(\psi) = \sum_{k=1}^p \delta_{kN}^{N+}(A_k^T \psi^0) + \sum_{k=1}^{p-1} \sum_{j=1}^N \delta_{kj}^{N+} (\psi^N(t_{kj}^{N-}) - \psi^N(t_{kj}^N))$$

$$+ \sum_{j=1}^{N-1} \delta_{pj}^{N+} (\psi^N(t_{pj}^{N-}) - \psi^N(t_{pj}^N)) - \delta_{pN}^{N+} (\psi^N(-h)).$$

Since (H3) is valid $\|(A_{01}^T \psi^0)^N - A_{01}^T \psi^0\|_2 \rightarrow 0$.

If $\psi \in D^*$, $|\psi^N(0) - \psi^1(0)| \leq \frac{\text{const}}{N^2} \|\ddot{\psi}^1\|_\infty$. Defining $\tilde{\psi}^1 = \sum_{k=1}^{p-1} A_k^T \psi^0 \chi_{[-h, -h_k]}$ we have $\tilde{\psi}^1 \in Y^N$ for all N , $D^+ \tilde{\psi}^1 = 0$ and $\psi^1 + \tilde{\psi}^1 \in W^{1,2}(-h, 0; \mathbb{R}^n)$ if $\psi \in \text{dom } A^*$. Hence

$$\|D^+ \psi^N - D(\psi^1 + \tilde{\psi}^1)\|_2 \leq (h)^{1/2} \|D^+(\psi^1 + \tilde{\psi}^1)^N - D(\psi^1 + \tilde{\psi}^1)\|_\infty \leq \frac{\text{const}}{N} \|\ddot{\psi}^1\|_\infty.$$

by Lemma 4.2. We arrange

$$\begin{aligned} \Delta^{N*}(\psi) &= [\delta_{pN}^{N+} (A_p^T \psi^0) - \delta_{pN}^{N+} (\psi^N(-h))] + \sum_{k=1}^{p-1} (\delta_{kN}^{N+} (A_k^T \psi^0) + \delta_{kN}^{N+} (\psi^N(t_{kN}^{N-}) - \psi^N(t_{kN}^N))) \\ &+ \sum_{k=1}^p \sum_{j=1}^{N-1} \delta_{kj}^{N+} (\psi^N(t_{kj}^{N-}) - \psi^N(t_{kj}^N)) = \Delta_1^{N*}(\psi) + \Delta_2^{N*}(\psi) + \Delta_3^{N*}(\psi). \end{aligned}$$

Recall, that for $\psi \in \text{dom } A^*$, $\psi^1(-h) = A_p^T \psi^0$ and $\psi^1(-h_k) - \psi^1(-h_k^-) = A_k^T \psi^0$, $k = 1, \dots, p-1$, so that

$$\|\Delta_1^{N*}(\psi)\|_2 = \|\delta_{pN}^{N+} (\psi^1(-h) - \psi^N(-h))\|_2 \leq 2 \left(\frac{N}{r_p}\right)^{1/2} \|\psi^1 - \psi^N\|_\infty \leq \frac{\text{const}}{N^{3/2}} \|\ddot{\psi}^1\|_\infty$$

and

$$\begin{aligned} \|\Delta_2^{N*}(\psi)\|_2 &= \left\| \sum_{k=1}^{p-1} \delta_{kN}^{N+} (\psi^1(-h_k) - \psi^1(-h_k^-) + \psi^N(-h_k^-) - \psi^N(-h_k)) \right\|_2 \\ &\leq 4 \left(\frac{N}{r}\right)^{1/2} (p-1) \|\psi^1 - \psi^N\|_\infty \leq \frac{\text{const}}{N^{3/2}} \|\ddot{\psi}^1\|_\infty. \end{aligned}$$

Finally, $\Delta_3^{N^*}(\psi)$ can be estimated like Δ_1^N in part (a), namely

$$\Delta_3^{N^*}(\psi) \leq \frac{\text{const}}{N} \|\ddot{\psi}^1\|_\infty, \psi \in D^*.$$



In order to apply Theorem 3.1 to A^N and $(A^N)^*$ of this section, it remains to show (H2) (i) and (H2*) (i).

Lemma 4.7: The sets $(\lambda-A)D$ and $(\lambda-A^*)D^*$ are dense in M^2 , if $\lambda > \omega$ as given in (2.4).

Proof: Due to basic facts in semigroup theory, we know from (2.5) that $\{\lambda \in \mathbb{C} \mid \text{Re } \lambda > \omega\}$ is contained in the resolvent sets $\rho(A)$ and $\rho(A^*)$. Hence, if $\lambda > \omega$, the operators $(\lambda-A)^{-1}$ and $(\lambda-A^*)^{-1}$ exist. Therefore, given $\psi \in \mathbb{R}^n \times C^1(-h, 0; \mathbb{R}^n)$ the equation $(\lambda-A)\phi = \psi$ has a unique solution $\phi = (\phi^1(0), \phi^1) \in \text{dom } A$, which by the definition of A satisfies $\lambda\phi^1 - \dot{\phi}^1 = \psi^1$. But this implies $\dot{\phi}^1 = \lambda\phi^1 - \psi^1$ is continuous and differentiable. In fact, $\ddot{\phi}^1 = \lambda\dot{\phi}^1 - \dot{\psi}^1$ is continuous, so that $\phi \in \mathbb{R}^n \times C^2(-h, 0; \mathbb{R}^n)$. Hence the dense set $\mathbb{R}^n \times C^1$ is contained in $(\lambda-A)(\mathbb{R}^n \times C^2 \cap \text{dom } A)$, which is a subset of $(\lambda-A)D$.

The same arguments applied to $\psi^1|_{I_k} \in C^1(I_k; \mathbb{R}^n)$, $k = 1, \dots, p$, reveal that $(\lambda-A^*)D^*$ is dense, if $A_{01} \in L(\mathbb{R}^n; C^1(-h, 0; \mathbb{R}^n))$. Let $\lambda > \omega$, $\phi \in M^2$, $0 < \epsilon < \|\phi\|$ and $A_{01} \in L(\mathbb{R}^n; L^2(-h, 0; \mathbb{R}^n))$. There is a $A_{01}^c \in L(\mathbb{R}^n; C^1(-h, 0; \mathbb{R}^n))$ with corresponding generator A_c^* satisfying $\|A^* - A_c^*\|_{L(M^2)} = \|A_{01} - A_{01}^c\|_{L(\mathbb{R}^n; L^2)} < \epsilon(4\|\phi\| \|(\lambda-A^*)^{-1}\|)^{-1}$, $\lambda \in \rho(A_c^*)$,

and $\|(\lambda - A_c^*)^{-1}\| \leq 2 \|(\lambda - A^*)^{-1}\|$ (cf. [15], IV. Th. 1.16). Let $\psi_c \in D^*$ such that $\|(\lambda - A_c^*)\psi_c - \phi\| < \varepsilon$. Then $\|(\lambda - A_c^*)\psi_c\| \leq 2\|\phi\|$ and $\|(\lambda - A^*)\psi_c - \phi\| \leq \|(\lambda - A_c^*)\psi_c - \phi\| + \|A_c^* - A^*\| \|\psi_c\| \leq \varepsilon + \varepsilon(4\|\phi\| \|(\lambda - A^*)^{-1}\|)^{-1} \|\psi_c\| < 2\varepsilon$, because $\|\psi_c\| \leq \|(\lambda - A_c^*)^{-1}\| \|(\lambda - A_c^*)\psi_c\| \leq 4\|(\lambda - A^*)^{-1}\| \|\phi\|$. This proves the density of $(\lambda - A^*)D^*$. ■

The proofs of the Lemmas 4.6, 4.7 do not use additional smoothness assumptions on A_{01} as it is the case in [12], [13], [14].

Summarizing, we can now say that the semigroups $S^N(\cdot)$ and $S^N(\cdot)^*$ generated by $A_{p^N}^N$ and $(A^N)^*_{p^N}$ converge to the hereditary semigroups $S(\cdot)$ and $S(\cdot)^*$ in the sense of Theorem 3.1. Also, Theorem 3.3 holds so that for the finite time horizon control problem we can approximate the Riccati operators and the optimal controls by solving the finite dimensional systems (3.4), (3.5).

4.3. Vector dominance preservation

With respect to the basis $\hat{e}_0, \hat{e}_{11}, \dots, \hat{e}_{p,2N}^N$ of Z^N and the canonical basis of \mathbb{R}^N the matrices $[A^N]$ and $[A^{N*}]$ representing A^N and A^{N*} are given by

$$(4.12) \quad [A^N] = (Q^N)^{-1} H^N \quad \text{and} \quad [A^{N*}] = (Q^N)^{-1} H^{NT}$$

with

$$H^N = \begin{bmatrix} \langle \hat{e}_0, A^N \hat{e}_0 \rangle & \dots & \langle \hat{e}_0, A^N \hat{e}_{p,2N}^N \rangle \\ \vdots & & \vdots \\ \langle \hat{e}_{p,2N}^N, A^N \hat{e}_0 \rangle & \dots & \langle \hat{e}_{p,2N}^N, A^N \hat{e}_{p,2N}^N \rangle \end{bmatrix}, \quad N = 1, 2, \dots$$

For the computation of the entries of H^N , observe that the derivatives of the basis elements are given by

$$\frac{d^+}{dt} e_{k,2j-1}^N = 0 \quad \text{and} \quad \frac{d^+}{dt} e_{k,2j}^N = \frac{2N}{r_k} e_{k,2j-1}^N, \quad k = 1, \dots, p, \quad j = 1, \dots, N.$$

Written in the simplifying $n \times n$ matrix notation we get for example

$$A^N e_{k,2j}^N = (A_{k,2j}^N, \frac{2N}{r_k} e_{k,2j-1}^N - \delta_{kj}^{N-1}(I) - \delta_{k,j-1}^{N-1}(I)),$$

$$k = 1, \dots, p, \quad j = 1, \dots, N-1,$$

with

$$A_{kj}^N = \int_{-h}^0 A_{01}(s) e_{kj}^N(s) ds, \quad k = 1, \dots, p, \quad j = 1, \dots, 2N.$$

The inner products in H^N are evaluated using the orthogonality of the basis elements and Proposition 4.1. The result is the $n(2pN+1)$ square matrix

$$H^N = \begin{bmatrix} A_0 & A_1^N & \dots & A_p^N \\ k_0 & h & & 0 \\ & k & & \\ & & 0 & \\ & & & k & h \end{bmatrix}$$

where

$$A_k^N = (A_{k1}^N \ A_{k2}^N \ \dots \ A_{k,2N-2}^N \ A_{k,2N-1}^N + A_k \ A_{k,2N}^N - A_k)$$

is a $n \times 2nN$ matrix, $k = 1, \dots, p$, $k_0 = \begin{pmatrix} I \\ I \end{pmatrix}$ a $2n \times n$ matrix and $h = \begin{pmatrix} -I & I \\ -I & -I \end{pmatrix}$, $k = \begin{pmatrix} I & -I \\ I & -I \end{pmatrix}$ are $2n \times 2n$ matrices. Numerical algorithms solving high order systems with coefficients $[A^N]$ might take considerable advantage from the fact that H^N has band structure (not the case for the Legendre methods [11], [12], [13]) and that Q^N is diagonal (not the case for spline methods [14]).

In the following, we exploit the structure of the matrix $[A^N]$ in order to deduce the VDP property and (H4) for our piecewise linear approximations.

Lemma 4.8: If there is a $c_2 \geq 0$ independent on N such that for all $\phi \in Z^N$

$$\int_0^\infty |VS^N(t)\phi|^2 dt \leq c_2 \|\phi\|^2,$$

then

$$\int_0^\infty \|S^N(t)\phi\|^2 dt \leq c_1 \|\phi\|^2, \quad \phi \in Z^N,$$

for some $c_1 \geq 0$ not depending on N or ϕ .

Proof: Let $\phi \in Z^N$ and set $S^N(t)\phi = e^{A^N P^N t} \phi = e^{A^N t} \phi = (w_0^N(t), w_1^N(t)) \in Z^N, t \geq 0$. With $w_1^N(t) = \sum_{k=1}^p \sum_{j=1}^{2N} e_{kj}^N w_{kj}^N(t)$ the coefficient vector $\text{col}(w_0^N(t), w_{11}^N(t), \dots, w_{p,2N}^N(t))$ is the solution of

$$(4.13) \quad \frac{d}{dt} \begin{bmatrix} w_0^N(t) \\ w_{11}^N(t) \\ \vdots \\ w_{p,2N}^N(t) \end{bmatrix} = [A^N] \begin{bmatrix} w_0^N(t) \\ w_{11}^N(t) \\ \vdots \\ w_{p,2N}^N(t) \end{bmatrix}, \quad t \geq 0, \quad \begin{matrix} w_0^N(0) = \phi^0 \\ w_{kj}^N(0) = \phi_{kj}^N \end{matrix}.$$

A view of the rows of $[A^N]$ reveals that (4.13) implies

$$(4.14) \quad \begin{aligned} \dot{v}_{11}^N(t) &= -\frac{N}{r_1} a v_{11}^N(t) + \frac{N}{r_1} v_0^N(t) \\ \dot{v}_{k1}^N(t) &= -\frac{N}{r_k} a v_{k1}^N(t) + \frac{N}{r_k} b v_{k-1,N}^N(t), \quad k = 2, \dots, p \\ \dot{v}_{kj}^N(t) &= -\frac{N}{r_k} a v_{kj}^N(t) + \frac{N}{r_k} b v_{k,j-1}^N(t), \quad k = 1, \dots, p, \quad j = 2, \dots, N \\ v_{kj}^N(0) &= \text{col}(\phi_{k,2j-1}^N, \phi_{k,2j}^N), \quad k = 1, \dots, p, \quad j = 1, \dots, N \end{aligned}$$

where $v_{kj}^N(t) = \text{col}(w_{k,2j-1}^N(t), w_{k,2j}^N(t)) \in \mathbb{R}^{2n}, v_0^N(t) = \text{col}(w_0^N(t), 3w_0^N(t))$ and $a = \begin{pmatrix} 1 & -1 \\ 3 & 3 \end{pmatrix} \otimes I, b = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \otimes I$. Using (4.6) we get

$$(4.15) \quad |v_{kj}^N(0)|^2 \leq \frac{4N}{r_k} \|\phi\|^2, \quad k = 1, \dots, p, \quad j = 1, \dots, N.$$

To estimate the solutions of (4.14), we make use of the fact that, if $f \in L^2(0, \infty; \mathbb{R}), \alpha > 0$ and $g(t) = \int_0^t e^{-\alpha(t-s)} f(s) ds, t \geq 0$, then

$g \in L^2(0, \infty; \mathbb{R})$ and $\int_0^\infty g^2(t) dt \leq \frac{1}{\alpha^2} \int_0^\infty f^2(t) dt$ (cf. for instance [10], VI. 2.33).

The first equation in (4.14) yields

$$v_{11}^N(t) = e^{-\frac{N}{r_1} t} \text{ at } v_{11}^N(0) + \int_0^t e^{-\frac{N}{r_1} a(t-s)} v_0^N(s) ds, \quad t \geq 0.$$

With $\tilde{v}_{11}^N(t) = v_{11}^N(t) - e^{-\frac{N}{r_1} t} \text{ at } v_{11}^N(0)$, $t \geq 0$, we get

$$\begin{aligned} \int_0^\infty |\tilde{v}_{11}^N(t)|^2 dt &\leq \frac{r_1^2}{N^2} \int_0^\infty e^{2|a|t} |v_0^N(t)|^2 dt \leq \text{const} \int_0^\infty |w_0^N(t)|^2 dt = \\ &= \text{const} \int_0^\infty |VS^N(t)\phi|^2 dt \leq \text{const} \|\phi\|^2, \end{aligned}$$

by assumption, where the constants do not depend on N . Using (4.15) it follows

$$(4.16) \quad \int_0^\infty |v_{11}^N(t)|^2 dt \leq \text{const} \|\phi\|^2.$$

Estimating the solutions of the other equations in (4.14) by the same method yields

$$\int_0^\infty |v_{k,j}^N(t)|^2 dt \leq \text{const} \|\phi\|^2 \quad \text{if} \quad \int_0^\infty |v_{k,j-1}^N(t)|^2 dt \leq \text{const} \|\phi\|^2,$$

$$k = 1, \dots, p, \quad j = 2, \dots, N$$

and

$$\int_0^\infty |v_{k1}^N(t)|^2 dt \leq \text{const } \|\phi\|^2 \quad \text{if} \quad \int_0^\infty |v_{k-1,N}^N(t)|^2 dt \leq \text{const } \|\phi\|^2, \quad k = 2, \dots, p.$$

So, from (4.16), we get by induction

$$\int_0^\infty |v_{kj}^N(t)|^2 dt \leq \text{const } \|\phi\|^2, \quad k = 1, \dots, p, \quad j = 1, \dots, N.$$

But this proves the assertion of the lemma, because by (4.3)

$$\begin{aligned} \|S^N(t)\phi\|^2 &= |w_0^N(t)|^2 + \sum_{k=1}^p \sum_{j=1}^N \left[\frac{r_k}{N} |w_{k,2j-1}^N(t)|^2 + \frac{r_k}{3N} |w_{k,2j}^N(t)|^2 \right] \\ &= |w_0^N(t)|^2 + \sum_{k=1}^p \sum_{j=1}^N \frac{r_k}{N} |dv_{kj}^N(t)|^2, \end{aligned}$$

where $d = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1/3} \end{pmatrix} \otimes I$

so that

$$\begin{aligned} \int_0^\infty \|S^N(t)\phi\|^2 dt &\leq \int_0^\infty |w_0^N(t)|^2 dt + \sum_{k=1}^p \frac{r_k}{N} \sum_{j=1}^N |d|^2 \int_0^\infty |v_{kj}^N(t)|^2 dt \\ &\leq \text{const } \|\phi\|^2. \end{aligned}$$

Corollary 4.9: The semigroups $\overline{S}^N(\cdot)$ generated by $\overline{A}^N_p = (A^N - B^N R^{-1} (B^N)^* \Pi^N)_p^N$ satisfy hypothesis (H4).

Proof: Since $B^N: \mathbb{R}^1 \rightarrow Z^N$ is represented by the $n(2pN+1) \times m$ matrix

$$(4.17) \quad [B^N] = \text{col}(B_0, 0, \dots, 0),$$

the matrix $[\bar{A}^N]$ differs from $[A^N]$ only in the first n rows. Therefore, (4.13) with $[A^N]$ replaced by $[\bar{A}^N]$ again yields (4.14). The rest of the proof remains unchanged. ■

Besides for (H4), the VDP-property of the piecewise linear approximations together with the results of the next section will be used to deduce (H5).

4.4 The eigenvalues of the approximate systems

It is known that the spectrum of the generator A coincides with its point spectrum namely $\sigma(A) = \{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0\}$ where

$$\Delta(\lambda) = \lambda I - \sum_{k=0}^p A_k e^{-\lambda h_k} - \int_{-h}^0 A_{01}(s) e^{\lambda s} ds, \quad \lambda \in \mathbb{C}.$$

The eigenvalues of the finite dimensional operators A^N are the zeros of $\det(\lambda I^N - [A^N])$ in \mathbb{C} . I^N denotes the $n(2pN+1)$ -identity matrix. In order to calculate the determinant of the characteristic matrix $\Delta^N(\lambda) = \lambda I^N - [A^N]$ in terms of rational functions of λ , we transform $\Delta^N(\lambda)$ by elementary row and column operations.

First, for all $k = 1, \dots, p$ and $j = 1, \dots, N$ multiply $\Delta^N(\lambda)$ from the right by the matrices

$$S_{kj}^N = \begin{bmatrix} I & & & 0 \\ & I & & \\ & & 0 & \\ & & & I \end{bmatrix},$$

the only nondiagonal identity matrix being at the position $((k, 2j-1), (k, 2j))$. The resulting matrix is multiplied from the left by

$$(4.18) \quad p_k^N(\lambda) = \frac{N}{r_k} - \frac{3N}{r_k} \lambda \left(\lambda + \frac{6N}{r_k} \right)^{-1}, \quad k = 1, \dots, p.$$

$$q_k^N(\lambda) = \lambda + p_k^N(\lambda)$$

We define

$$D_{pN}^N = (-B_{pN}^N)(q_p^N(\lambda))^{-1}$$

$$(4.19) \quad D_{kj}^N = (-B_{kj}^N)(q_k^N(\lambda))^{-1} + D_{k,j+1}^N p_k^N(\lambda)(q_k^N(\lambda))^{-1}, \quad k = 1, \dots, p, \quad j = 1, \dots, N-1$$

$$D_{kN}^N = (-B_{kN}^N)(q_k^N(\lambda))^{-1} + D_{k+1,1}^N p_{k+1}^N(\lambda)(q_k^N(\lambda))^{-1}, \quad k = 1, \dots, p-1.$$

Multiplication from the left with

$$D^N = \begin{bmatrix} I & D_{11}^N 0 & D_{12}^N 0 & \dots & D_{pN}^N 0 \\ & 0 & & & 0 \\ & & & & I \end{bmatrix}$$

subtracts $p_1^N(\lambda)D_{11}^N$ from the upper left block and sets all other entries in the first row of blocks to zero. Thus

$$(4.20) \quad \det \Delta^N(\lambda) = \left[\prod_{k=1}^p (\det q_k^N(\lambda)I)^N (\det(\lambda + \frac{6N}{r_k})I)^N \right] \det \Delta_0^N(\lambda),$$

where

$$(4.21) \quad \Delta_0^N(\lambda) = \lambda I - A_0 - \frac{3N}{r_1} C_{12}^N - p_1^N(\lambda)D_{11}^N.$$

Note that these transformations are possible, as far as $\lambda + \frac{6N}{r_k} \neq 0$ and $q_k^N(\lambda) \neq 0$, i.e., $\lambda \neq -\frac{6N}{r_k}$ and $\lambda \neq \frac{2N}{r_k} \pm i\frac{N}{r_k}\sqrt{2}$, $k = 1, \dots, p$, in particular if $\lambda \in \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq -\rho\}$ and $N \geq \frac{2\rho}{r}$ for some $\rho > 0$. Further note, if $\operatorname{Re} \lambda \geq -\rho$, then $\det \Delta^N(\lambda) = 0$ iff $\det \Delta_0^N(\lambda) = 0$, provided $N \geq \frac{2\rho}{r}$. From (4.19) we get by recursion

$$p_k^N(\lambda) D_{k1}^N = \sum_{m=k+1}^p \left(\sum_{\ell=1}^N (-B_{m\ell}^N)(r_m^N(\lambda))^\ell \right) \prod_{i=k}^{m-1} (r_i^N(\lambda))^N + \sum_{\ell=1}^N (-B_{k\ell}^N)(r_k^N(\lambda))^\ell,$$

$$k = 1, \dots, p$$

where

$$(4.22) \quad r_k^N(\lambda) = p_k^N(\lambda)(q_k^N(\lambda))^{-1}, \quad k = 1, \dots, p.$$

Using this in (4.21) yields

$$\Delta_0^N(\lambda) = \lambda I - L^N(\lambda) = \lambda I - (L_1^N(\lambda) + L_2^N(\lambda) + L_3^N(\lambda))$$

with

$$L_1^N(\lambda) = A_0 + \sum_{k=1}^p \prod_{i=1}^k (r_i^N(\lambda))^N A_k$$

$$L_2^N(\lambda) = \sum_{k=1}^p \sum_{i=1}^{k-1} (r_i^N(\lambda))^N \sum_{j=1}^N A_{k,2j-1}^N (r_k^N(\lambda))^{-j}$$

$$(4.23) \quad L_3^N(\lambda) = \sum_{k=1}^p \frac{3N}{r_k} \left(\lambda + \frac{6N}{r_k}\right)^{-1} \prod_{i=1}^{k-1} (r_i^N(\lambda))^N \sum_{j=1}^N (A_{k,2j-1}^N + A_{k,2j}^N) ((r_k^N(\lambda))^{j-1} - (r_k^N(\lambda))^{-j}).$$

Proposition 4.10. (a) With the projection $V: M^2 \rightarrow \mathbb{R}^n$ introduced in (H4)
we have

$$[V(\lambda I - A^N)^{-1}V^*] = (\Delta_0^N(\lambda))^{-1}, \quad \lambda \in \rho(A^N).$$

(b) $(r_k^N(\lambda))^N \rightarrow e^{-r_k \lambda}$ and $r_k^N(\lambda) \rightarrow 1, \quad k = 1, \dots, p$

as $N \rightarrow \infty$, uniformly in λ on bounded subsets of \mathbb{C} .

(c) If $\text{Re } \lambda \geq 0$, then $|r_k^N(\lambda)| \leq 1, \quad k = 1, \dots, p$. For any $\rho > 0$
there exists $N(\rho)$ such that for all $N \geq N(\rho)$ and all $\lambda \in \mathbb{C}$ with
 $\text{Re } \lambda \in [-\rho, 0]$

$$|r_k^N(\lambda)| \leq |r_k^N(-\rho)|, \quad k = 1, \dots, p.$$

Proof: Expanding the determinant of $\Delta^N(\lambda)$ by elementary operations, we have seen that $T^N \Delta^N(\lambda) S^N = U^N(\lambda)$, where $U^N(\lambda)$ is a lower triangular block-matrix with $\Delta_0^N(\lambda)$ in the position (1,1) and S^N, T^N are regular matrices given by $S^N = \prod_{k=1}^p \prod_{j=1}^N S_{kj}^N$ and $T^N = D^N \prod_{k=1}^p \prod_{j=1}^N T_{kj}^N C^N$. The first column of blocks in T^N and the first row of blocks in $(S^N)^{-1}$ are of the form $(I \ 0 \ \dots \ 0)$. Thus, the application of these transformations to the equation $\Delta^N(\lambda)(\psi^0 * \dots *)^T = (\phi^0 \ 0 \ \dots \ 0)^T$ yields

$$\begin{aligned} U^N(\lambda)(\psi^0 * \dots *)^T &= T^N \Delta^N(\lambda) S^N (S^N)^{-1} (\psi^0 * \dots *)^T = \\ &= T^N (\phi^0 \ 0 \ \dots \ 0)^T = (\phi^0 \ 0 \ \dots \ 0)^T \end{aligned}$$

or

$$(\psi^0 * \dots *)^T = (U^N(\lambda))^{-1} (\phi^0 \ 0 \ \dots \ 0)^T \quad \text{for all } \phi^0, \psi^0 \in \mathbb{R}^n.$$

This implies $\psi^0 = (\Delta_0^N(\lambda))^{-1} \phi^0$. But

$$\begin{aligned} V(\lambda I - A^N)^{-1} V^* \phi^0 &= V(\lambda I - A^N)^{-1} (\phi^0, 0) = [V](\Delta_0^N(\lambda))^{-1} (\phi^0, 0) \\ &= [V](\psi^0, 0) = \psi^0 = (\Delta_0^N(\lambda))^{-1} \phi^0, \end{aligned}$$

for all $\phi^0 \in \mathbb{R}^n$, and this proves (a).

From (4.18), (4.22) we have

$$r_k^N(\lambda) = \frac{6N^2 - 2Nr_k\lambda}{6N^2 + 4Nr_k\lambda + \lambda^2 r_k^2} = \left(1 + \frac{r_k\lambda}{N}\right)^{-1} \left(1 + \frac{3r_k^2\lambda^2}{6N^2 + 4Nr_k\lambda - 2r_k^2\lambda^2}\right)^{-1},$$

$$k = 1, \dots, p.$$

This proves (b), because, as $N \rightarrow \infty$

$$\left(1 + \frac{r_k\lambda}{N}\right)^{-N} \rightarrow e^{-r_k\lambda} \quad \text{and} \quad \left(1 + \frac{3r_k^2\lambda^2}{6N^2 + 4Nr_k\lambda - 2r_k^2\lambda^2}\right)^{-N} \rightarrow 1$$

uniformly in λ on bounded subsets of \mathcal{D} .

Finally, an explicit computation of $|r_k^N(\lambda)|^2$ and its derivative with respect to $\text{Re}\lambda$ shows (c). ■

The properties (b) and (c) are essential for the convergence of the matrices $\Delta_0^N(\lambda)$.

(b) If $\det \Delta_0^N(\lambda) = 0$ and $\operatorname{Re} \lambda \geq 0$, then $|\lambda| \leq \sum_{k=0}^p |A_k| + 3 \|A_{01}\|_{L^1}$.

For any $\rho > 0$ there is an $N(\rho)$ such that for all λ with

$\operatorname{Re} \lambda \in [-\rho, 0]$ and all $N \geq N(\rho)$, $\det \Delta_0^N(\lambda) = 0$ implies

$$|\lambda| \leq \sum_{k=0}^p 2^k e^{\rho h k} |A_k| + 20 e^{\rho h} \|A_{01}\|_{L^1}.$$

Proof: Set $\Delta_0^N(\lambda) = \lambda I - L_1^N(\lambda) - L_2^N(\lambda) - L_3^N(\lambda)$ as in (4.23).

$$L_1^N(\lambda) \rightarrow A_0 + \sum_{k=1}^p A_k \prod_{i=1}^k e^{-r_i \lambda} = \sum_{k=0}^p e^{-h_k \lambda} A_k$$

and

$$\lim_{N \rightarrow \infty} L_2^N(\lambda) = \sum_{k=1}^p e^{-h_{k-1} \lambda} \lim_{N \rightarrow \infty} \sum_{j=1}^N \int_{-h}^0 A_{01}(s) e_{k, 2j-1}^N(s) (r_k^N(\lambda))^j ds$$

uniformly in λ on bounded sets.

Let $s \in I_k$. To each N there is a unique j_N such that $s \in I_{k, j_N}^N$;

thus

$$\lim_{N \rightarrow \infty} L_2^N(\lambda) = \sum_{k=1}^p e^{-h_{k-1} \lambda} \lim_{N \rightarrow \infty} \int_{I_k} A_{01}(s) (r_k^N(\lambda))^{j_N} ds.$$

From $0 \leq j_N + N(h_{k-1} + s)/r_k \leq 1$ and Proposition 4.10 (b), it follows

$$\begin{aligned} \lim_{N \rightarrow \infty} [(r_k^N(\lambda))^{-N(h_{k-1} + s)/r_k} - (r_k^N(\lambda))^{j_N}] &= \\ &= \lim_{N \rightarrow \infty} (r_k^N(\lambda))^{-N(h_{k-1} + s)/r_k} (1 - (r_k^N(\lambda))^{j_N + N(h_{k-1} + s)/r_k}) = 0; \end{aligned}$$

hence

$$\lim_{N \rightarrow \infty} (r_k^N(\lambda))^{j_N} = \lim_{N \rightarrow \infty} (r_k^N(\lambda))^{-N(h_{k-1} + s)/r_k} = e^{(h_{k-1} + s)\lambda}.$$

The convergence is uniform in λ on bounded sets and in $s \in I_k$ and dominated with respect to s . Therefore

$$\lim_{N \rightarrow \infty} L_2^N(\lambda) = \sum_{k=1}^p e^{-h_{k-1}\lambda} \int_{I_k} A_{01}(s) e^{(h_{k-1}+s)\lambda} ds = \int_{-h}^0 A_{01}(s) e^{\lambda s} ds.$$

For $L_3^N(\lambda)$ in (4.23) we get

$$|L_3^N(\lambda)| \leq \sum_{k=1}^p \left| \frac{3N}{r_k} \left(\lambda + \frac{6N}{r_k} \right)^{-1} \right| \prod_{i=1}^{k-1} |r_i^N(\lambda)|^N \int_{I_k} |A_{01}(s) (r_k^N(\lambda))^{j_N}| ds \left| (r_k^N(\lambda))^{-1} - 1 \right|$$

$(r_k^N(\lambda))^{-1}$ is defined if $\frac{3N}{r_k} \neq \lambda$, $k = 1, \dots, p$, so that $L_3^N(\lambda) \rightarrow 0$ uniformly on bounded sets. If $\det \Delta_0^N(\lambda) = 0$ then λ is an eigenvalue of $L^N(\lambda)$. Therefore $\det \Delta_0^N(\lambda) = 0$ implies $|\lambda| \leq |L_1^N(\lambda)| + |L_2^N(\lambda)| + |L_3^N(\lambda)|$. So (b) is proved by similar arguments as (a) using Proposition (4.10) (c) and the convergence of $r_k^N(-\rho)$, $(r_k^N(-\rho))^N$ and $(r_k^N(-\rho))^{j_N}$. ■

For the investigation of the asymptotic behavior of the eigenvalues of A^N , $N \rightarrow \infty$, we now use the following consequence of Rouché's Theorem:

Proposition 4.12. Let $f, f^N, N = 1, 2, \dots$ be holomorphic inside and on a closed bounded contour $\Gamma \neq \emptyset$. If f has no zeros on Γ and if $f^N \rightarrow f$ uniformly on Γ , then there exists an $N_0 \in \mathbb{N}$ such that for $N \geq N_0$, f^N and f have the same number of zeros (counted according to their multiplicities) inside Γ .

For $\rho > 0$ define $G_\rho = \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq -\rho\}$. Since $\det \Delta(\lambda)$ is analytic on G_ρ and its zeros lie in $\{\lambda \in \mathbb{C} \mid -\rho \leq \operatorname{Re} \lambda \leq 0 \text{ and } |\lambda| \leq \sum_{k=0}^p e^{\rho h_k} |A_k| + e^{\rho h} \|A_{01}\|_{L^1}\} \cup \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq 0 \text{ and } |\lambda| \leq \sum_{k=0}^p |A_k| + \|A_{01}\|_{L^1}\}$, the number of eigenvalues of the generator A in G_ρ is finite. Similarly, from Lemma 4.11 (b) and the notes following (4.21) we see that, if N is sufficiently large, the eigenvalues of A^N in G_ρ lie within

$$K_\rho = \{\lambda \in \mathbb{C} \mid -\rho \leq \operatorname{Re} \lambda \leq 0 \text{ and } |\lambda| \leq \sum_{k=0}^p 2^k e^{\rho h_k} |A_k| + 20e^{\rho h} \|A_{01}\|_{L^1}\} \\ \cup \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq 0 \text{ and } |\lambda| \leq \sum_{k=0}^p |A_k| + 3 \|A_{01}\|_{L^1}\}.$$

Although these estimates are not accurate, they lead to rigorous results given in

Lemma 4.13: (a) If λ_0 is an eigenvalue of A with multiplicity k , then for any $\varepsilon > 0$ (small enough) there is an N_0 such that each A^N , $N \geq N_0$, possesses k eigenvalues in $B(\lambda_0, \varepsilon) = \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_0| < \varepsilon\}$.

(b) Let $\rho > 0$ and $\lambda_i, i = 1, \dots, \ell$ be the eigenvalues of A in G_ρ . For any $\varepsilon > 0$ (small enough) there exists N_0 such that the operators A^N , $N \geq N_0$, have no eigenvalues in $G_\rho \setminus \bigcup_{i=1}^{\ell} B(\lambda_i, \varepsilon)$.

Proof: Assume that λ_0 is the only zero of $\det \Delta(\lambda)$ in $\overline{B(\lambda_0, \epsilon)}$. $\partial B(\lambda_0, \epsilon)$ is bounded and $\det \Delta_0^N(\lambda) \rightarrow \det \Delta(\lambda)$ uniformly on bounded sets. Thus, (a) follows at once from Proposition 4.12.

Choose, without loss of generality, $\epsilon > 0$ such that $\det \Delta(\lambda)$ has no zero in $\partial G_\rho \cup \bigcup_{i=1}^l \partial B(\lambda_i, \epsilon)$. We know from (4.20) that, if $N \geq \frac{2\rho}{\underline{r}}$, $\lambda \in G_\rho$ is an eigenvalue of A^N iff $\det \Delta_0^N(\lambda) = 0$. Write $G_\rho \setminus \bigcup_{i=1}^l B(\lambda_i, \epsilon) = G_1 \cup G_2$, where

$$G_1 = (G_\rho \cap K_\rho) \setminus \bigcup_{i=1}^l B(\lambda_i, \epsilon), \quad G_2 = (G_\rho \cap K_\rho^c) \setminus \bigcup_{i=1}^l B(\lambda_i, \epsilon).$$

G_1 is bounded and ∂G_1 contains no zero of $\det \Delta(\lambda)$. Thus, there is an $N_1 \geq \frac{2\rho}{\underline{r}}$ such that for all $N \geq N_1$ $\det \Delta_0^N(\lambda)$ has as many zeros in G_1 as $\det \Delta(\lambda)$, that is $\det \Delta_0^N(\lambda)$ has no zero in G_1 . Since $G_2 \subseteq K_\rho^c$, there is no eigenvalue of A^N in G_2 , if N is sufficiently large. ■

This shows that the eigenvalues of A are approximated by the eigenvalues of the operators A^N . Moreover, given $\epsilon > 0$, let $\rho \in \mathbb{R}$ and $\lambda_i, i = 1, \dots, l$, be the eigenvalues of the hereditary system with $\operatorname{Re} \lambda_i \geq \rho$. Then the piecewise linear approximations do not have eigenvalues in the right half plane $\operatorname{Re} \lambda \geq \rho$ outside the balls $B(\lambda_i, \epsilon), i = 1, \dots, l$, provided N is sufficiently large.

4.5. Uniform stability

From the results of the previous section, we conclude that, if $S(\cdot)$ is stable, i.e., $\|S(t)\| \leq Me^{-\omega_0 t}$ with some $M, \omega_0 > 0$, then for all $\omega < \omega_0$ there are an N_ω and constants M_N such that for all $N \geq N_\omega$

$$(4.24) \quad \|S^N(t)\| \leq M_N e^{-\omega t}.$$

In order to get uniformity with respect to N on the right hand side of these estimates, we follow an idea given by K. Ito [12] in connection with his Legendre-tau approximations. The idea is to establish uniformity in (4.24) for one special no-delay case and to interpret this special case as a perturbation of the general situation.

Let us consider the equation $\dot{x}(t) = -x(t), t \geq 0$, in \mathbb{R}^1 as if it were a functional differential equation with p delays, demanding the initial condition $(x(0), x_0) = \phi \in M^2$. We approximate by our piecewise linear scheme, denoting the approximating generators by A_0^N . They are given in Definition 4.3 where $A_0 = -I, A_k = 0, k = 1, \dots, p$ and $A_{01}(\cdot) \equiv 0$. The representation $[A_0^N]$ is a lower triangular block matrix with $-I$ in the left upper position. Therefore the first row of blocks in $e^{[A_0^N]t}$ is given by $(e^{-t}I \ 0 \ \dots \ 0)$. Hence, for all $\phi \in Z^N$

$$|Ve^{A_0^N t} \phi| = |\alpha_0(e^{[A_0^N]t} \alpha^N(\phi))| = |e^{-t} \alpha_0(\phi)| = e^{-t} |\phi^0| \leq e^{-t} \|\phi\|.$$

Thus, by Lemma 4.8, there is a $c > 0$ such that

$$(4.25) \quad \int_0^\infty \|e^{A_0^N t} \phi\|^2 dt \leq c \|\phi\|^2, \phi \in Z^N$$

and by Lemma 3.4 there exist constants $M_0, \alpha_0 > 0$ such that for all $N = 1, 2, \dots$

$$\|e^{A_0^N t} \Big|_{Z^N}\| \leq M_0 e^{-\alpha_0 t}, \quad t \geq 0.$$

Remark: Guided by these arguments, one easily sees that the spline approximation scheme presented in [14] does not have the VDP-property. Because the first row of blocks of the matrix representing the spline generators $s_{A_0^N}^N$ is also of type $(-I \ 0 \ \dots \ 0)$ one gets for all ϕ in the spline subspace $s_{Z^N}^N$, $|(e^{s_{A_0^N}^N t} \phi)^0| \leq e^{-t} \|\phi\|$, as above. Thus, if a spline analog of Lemma 4.8 holds, then by Lemma 3.4 $\|e^{s_{A_0^N}^N t} \Big|_{s_{Z^N}^N}\| \leq K e^{-\epsilon t}$, for some $K, \epsilon > 0$, independent of N . But this contradicts the peculiar eigenvalue behavior of the spline scheme (see [14], Prop. 5.13).

Lemma 4.14: If $\|S(t)\| \leq M e^{-\omega_0 t}$, $t \geq 0$ for some $M, \omega_0 > 0$ then for
all $\omega < \omega_0$ there exist N_ω and \tilde{M} such that for all $N \geq N_\omega$

$$\|S^N(t) \Big|_{Z^N}\| \leq \tilde{M} e^{-\omega t}, \quad t \geq 0.$$

Proof: Let $0 < \omega < \omega_0$. We have seen in Lemma 4.11 that there is a constant c such that $|L^N(-\omega + i\tau)| \leq c$, $\tau \in \mathbb{R}$, if $N \geq N(\omega)$. It follows

$$|(\Delta_0^N(-\omega + i\tau))^{-1}| \leq \frac{1}{|-\omega + i\tau| - c} \quad \text{for } |-\omega + i\tau| > c$$

($\det \Delta_0^N(-\omega + i\tau) \neq 0$ if N is large enough). Thus, from the uniform convergence of $\Delta_0^N(\lambda)$ to $\Delta(\lambda)$ on the set $\{\lambda = -\omega + i\tau \mid |\lambda| \leq c\}$, we have

$$(4.26) \quad |(\Delta_0^N(-\omega+i\tau))^{-1}| \leq \gamma, \tau \in \mathbb{R}, N \geq N_\omega$$

with some $N_\omega \in \mathbb{N}$ and $\gamma > 0$.

Define $A_\omega^N = \omega I + A^N$ and let $\phi \in Z^N$. The trajectory $e^{A_0^N t} \phi$ is the solution of

$$\dot{z}(t) = A_0^N z(t) = A_\omega^N z(t) + (A_0^N - A_\omega^N)z(t), \quad t \geq 0, \quad z(0) = \phi.$$

Equivalently,

$$e^{A_0^N t} \phi = e^{A_\omega^N t} \phi + \int_0^t e^{A_\omega^N(t-s)} (A_0^N - A_\omega^N) e^{A_0^N s} \phi ds,$$

hence

$$e^{A_\omega^N t} \phi = e^{A_0^N t} \phi + \int_0^t e^{A_\omega^N(t-s)} V^* f^N(s, \phi) ds,$$

where $f^N(\cdot, \phi): \mathbb{R} \rightarrow \mathbb{R}^n$ is given by $f^N(s, \phi) = Fe^{A_0^N s} \phi$, $s \geq 0$ with $F: M^2 \rightarrow \mathbb{R}^n$.

$$(4.27) \quad F\psi = (I + \omega I + A_0) \psi^0 + \sum_{k=1}^p A_k \psi^1(-h_k) + \int_{-h}^0 A_{01}(s) \psi^1(s) ds,$$

because the L^2 -component of $(A_\omega^N - A_0^N)\psi$ vanishes for all $\psi \in Z^N$. From (4.27) and (4.25) it is clear that

$$(4.28) \quad \|f^N(\cdot, \phi)\|_{L^2(0, \infty; \mathbb{R}^n)} \leq \kappa \|\phi\|_{M^2}$$

for some $\kappa > 0$ not depending on N . We write

$$(4.29) \quad \text{Ve}^{\omega^N t} \phi = \text{Ve}^{A^N t} \phi + y(t), \quad t \geq 0$$

with

$$y(t) = \int_0^t \text{Ve}^{\omega^N(t-s)} V^* f^N(s, \phi) ds, \quad t \geq 0.$$

Letting $f^N(s, \phi) = 0, s \leq 0$ and $e^{\omega^N(t-s)} = 0, s > t$ or $s \leq 0$, we have $f^N(\cdot, \phi) \in L^1(\mathbb{R}; \mathbb{R}^n) \cap L^2(\mathbb{R}; \mathbb{R}^n)$ and by (4.24) $e^{\omega^N \cdot} \in L^1(\mathbb{R}; L(Z^N)) \cap L^2(\mathbb{R}; L(Z^N))$, if N is sufficiently large. The calculation of the Fourier-transform $\hat{y}(\cdot)$ of the convolution $y(\cdot)$ yields [10, (21.41)]

$$\begin{aligned} \hat{y}(\tau) &= (\text{Ve}^{\omega^N \cdot} V^*) \hat{(\tau)}(f^N(\cdot, \phi)) \hat{(\tau)} = \\ &= \int_{-\infty}^{\infty} e^{-i\tau t} \text{Ve}^{\omega^N t} V^* dt (f^N(\cdot, \phi)) \hat{(\tau)} = \\ &= V(-\omega + i\tau - A^N)^{-1} V^* (f^N(\cdot, \phi)) \hat{(\tau)}, \quad \tau \in \mathbb{R}. \end{aligned}$$

By Plancherel's Theorem (cf. for instance [10, (21.53)]) and (4.28), we get

$$\int_{-\infty}^{\infty} |(f^N(\cdot, \phi)) \hat{(\tau)}|^2 d\tau = \int_0^{\infty} |f^N(t, \phi)|^2 dt \leq \kappa^2 \|\phi\|^2.$$

Thus

$$\begin{aligned} \int_0^{\infty} |y(t)|^2 dt &= \int_{-\infty}^{\infty} |\hat{y}(\tau)|^2 d\tau \leq \\ &\leq \int_{-\infty}^{\infty} |V(-\omega + i\tau - A^N)^{-1} V^*|^2 |(f^N(\cdot, \phi)) \hat{(\tau)}|^2 d\tau = \end{aligned}$$

$$= \int_{-\infty}^{\infty} |(\Delta_0^N(-\omega+i\tau))^{-1}|^2 |(\hat{f}^N(\cdot, \phi))(\tau)|^2 d\tau$$

by Proposition 4.10 (a). Therefore, from (4.26)

$$\int_0^{\infty} |y(t)|^2 dt \leq \gamma^2 \kappa^2 \|\phi\|^2, \quad N \geq N_{\omega}$$

and, by (4.29), (4.25) and Lemma 4.8

$$\int_0^{\infty} \|e^{A_{\omega}^N t} \phi\|^2 dt \leq \text{const} \|\phi\|^2, \quad N \geq N_{\omega}, \phi \in Z^N.$$

Lemma 3.4 thus yields constants $\tilde{M}, \epsilon > 0$ not depending on N , such that

$$e^{\omega t} \|e^{A_{\omega}^N t}\|_{Z^N} = \|e^{A_{\omega}^N t}\|_{Z^N} \leq \tilde{M} e^{-\epsilon t}, \quad N \geq N_{\omega}. \quad \blacksquare$$

So, if the hereditary semigroup $S(\cdot)$ is exponentially stable with some decay rate ω_0 , it is approximated by piecewise linear systems with decay rate arbitrarily close to ω_0 .

Another immediate consequence of Lemma 4.14 is

Corollary 4.15: The semigroups $\tilde{S}^N(\cdot)$ generated by $\tilde{A}_{p^N}^N = (A^N - BR^{-1}B^* \Pi)p^N$ satisfy hypothesis (H5).

Proof: If the hereditary system is stabilizable, there exists a solution Π of the algebraic Riccati equation, and the closed loop semigroup $\bar{S}(\cdot)$ generated by $\bar{A} = A - BR^{-1}B^* \Pi$ is exponentially stable. The approximations to $\bar{S}(\cdot)$ are actually given by $\tilde{S}^N(\cdot)$. Thus the previous

lemma yields constants $\tilde{M}, \omega > 0$ such that

$$\|\tilde{S}^N(t)\|_{Z^N} \leq \tilde{M}e^{-\omega t}, \quad t \geq 0$$

for all N sufficiently large. ■

Summarizing, we have proved that the convergence statements of Theorem 3.5 are valid when the piecewise linear approximation scheme is applied to infinite time horizon hereditary control problems.

4.6. Examples

Testing the numerical performance of the approximation scheme developed in this paper, we have employed it in several examples and compared the outcomes with the results produced by other schemes. As far as these examples are representative, it turned out that the piecewise linear and the first order spline approximations [14] are of the same numerical accuracy (for the same approximation index N), but both are inferior to the Legendre methods [11,13].

In the case of the finite time horizon the Riccati differential equation

$$\begin{aligned} \frac{d}{dt} [\Pi^N(t)] + [(A^N)^*] [\Pi^N(t)] + [\Pi^N(t)] [A^N] - \\ - [\Pi^N(t)] [B^N] R^{-1} [B^N]^T [\Pi^N(t)] + [W^N] = 0, \quad 0 \leq t \leq T, \end{aligned}$$

$$[\Pi^N(T)] = [G^N]$$

is transformed, by taking $\Gamma^N(t) = Q^N[\Pi^N(T-t)]$, to a standard Riccati matrix differential equation

$$(4.30) \quad \frac{d}{dt} \Gamma^N(t) + [A^N]^T \Gamma^N + \Gamma^N [A^N] - \\ - \Gamma^N [B^N] R^{-1} [B^N]^T \Gamma^N + [W^N] = 0, \quad 0 \leq t \leq T$$

$$\Gamma^N(0) = [G^N].$$

Observe that the selfadjointness of $\Pi^N(t)$ implies $[\Pi^N(t)]^T Q^N = Q^N [\Pi^N(t)]$ and hence $\Gamma^N(t)^T = \Gamma^N(t)$. Since B^N, W^N, G^N refer exclusively to the \mathbb{R}^n -component of Z^N , we have

$$[B^N] = \text{col}(B_0, 0, \dots, 0) \in \mathbb{R}^{n(2pN+1) \times n}$$

$$[W^N] = \begin{bmatrix} W_0 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n(2pN+1) \times n(2pN+1)}$$

$$[G^N] = \begin{bmatrix} G_0 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n(2pN+1) \times n(2pN+1)}.$$

Thus (if $p = 1, A_{01} \equiv 0$) we can reduce the dimension of (4.30) introducing the $2n \times n(2N+1)$ matrices

$$F_1^N = \begin{bmatrix} F_0 & 0 & \text{-----} & 0 & G_0^T A_1 & - G_0^T A_1 \\ A_1^T G_0 & 0 & \text{-----} & 0 & & \end{bmatrix}, \quad F_2^N = \begin{bmatrix} I & 0 & \text{-----} & 0 \\ 0 & \text{-----} & 0 & I - I \end{bmatrix}$$

where F_0 is given by

$$F_0 = A_0^T G_0 + G_0^T A_0 - G_0^T B_0 R^{-1} B_0^T G_0 + W_0,$$

in order to get the factorization

$$\dot{\Gamma}(0) = (F_1^N)^T (F_2^N).$$

This implies ([18], p. 304 ff.)

$$(4.31) \quad \Gamma^N(t) = [G^N] + \int_0^t L_1^N(s)^T L_2^N(s) ds,$$

$L_i^N(t)$ being the solution of

$$(4.32) \quad \frac{d}{dt} L_i^N(t) = L_i^N(t) ([A^N] - [B^N] R^{-1} [B^N]^T \Gamma^N(t))$$

$$L_i^N(0) = F_i^N, \quad i = 1, 2.$$

Multiplying (4.31) from the left with $[B^N]^T$ yields

$$(4.33) \quad [B^N]^T \Gamma^N(t) = [B^N]^T L_1^N(t)^T L_2^N(t), \quad 0 \leq t \leq T$$

$$[B^N]^T \Gamma^N(0) = [B^N]^T [G^N].$$

Solving the $n(4n+m)(2N+1)$ differential equations (4.32), (4.33) we get $[B^N]^T \Gamma^N(t)$. But this is all we need for the computation of the suboptimal control $\hat{u}^N(t)$ (see (3.6)). Denoting the $m \times n$ blocks in $[B^N]^T [\Pi^N(t)]$ by $\beta_0^N(t), \dots, \beta_{2N}^N(t)$, we have

$$(4.34) \quad \hat{u}^N(t) = -R^{-1} \{ \beta_0^N(t) \hat{x}^N(t) + \sum_{j=1}^{2N} \int_{-h}^0 \beta_j^N(t) e_{1j}^N(s) \hat{x}^N(t+s) ds \}, \quad 0 \leq t \leq T,$$

where $\hat{x}^N(t)$ is the solution of

$$(4.35) \quad \dot{\hat{x}}(t) = A_0 \hat{x}(t) + A_1 \hat{x}(t-h) + B_0 \hat{u}^N(t)$$

in \mathbb{R}^n . In each term of the sum in (4.34), the integration ranges only over one of the intervals I_{1j}^N , $j = 1, \dots, N$.

Numerically the systems (4.34), (4.35) were solved simultaneously by an appropriately adjusted 4th order Runge-Kutta procedure combined with Simpson's rule for the evaluation of the integrals.

Example 4.1. Minimize

$$J(u) = \frac{1}{2} (x_1(2)^2 + x_2(2)^2) + \frac{1}{2} \int_0^2 (u_1(t)^2 + u_2(t)^2) dt$$

subject to

$$\dot{\hat{x}}(t) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \hat{x}(t-1) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u(t), \quad 0 \leq t \leq 2,$$

$$\hat{x}(0) = \hat{x}_0(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad -1 \leq t \leq 0.$$

The true solutions $\bar{u}(t), \bar{x}(t)$ (see [2]) and the piecewise linear approximations with index $N = 4, 8, 16$ are presented in the Tables 4.1 and 4.2. The relatively greatest errors occur around $t = 1$ and $t = 2$, where the derivatives of $\bar{x}(t)$ and $\bar{u}(t)$ have jumps, while $\hat{x}^N(t)$ and $\hat{u}^N(t)$ are of course continuously differentiable.

For the infinite time horizon control problem ($p = 1, A_{01} \equiv 0$) the suboptimal control and trajectory are again calculated via (4.34), (4.35) when $\Pi^N(t)$ is replaced by the stationary operator Π^N , that is the solution of the algebraic Riccati equation (3.7). The transformation $\Gamma^N = Q^N[\Pi^N]$ yields a standard Riccati matrix equation

$$[A^N]^T \Gamma^N + \Gamma^N [A^N] - \Gamma^N [B^N] R^{-1} [B^N]^T \Gamma^N + [W^N] = 0$$

in $\mathbb{R}^{n(2N+1) \times n(2N+1)}$, which was solved by the Newton-Kleinman-algorithm as presented in [18]. In each step of this algorithm, a Ljapunov matrix equation was solved using the quadratic procedure given by R. A. Smith (see also [18], p. 297). The time independent $m \times n$ blocks $\beta_j^N, j = 0, \dots, 2N$ were then employed in (4.35). Furthermore, with

$$[\Pi^N] = \begin{bmatrix} \Pi_{00}^N & \Pi_{11}^N & \dots & \Pi_{1,2N}^N \\ * & \text{---} & & * \\ | & & & | \\ * & \text{---} & & * \end{bmatrix}$$

we give some values of the feedback kernel

$$\Pi_1^N(s) = \sum_{j=1}^{2N} \Pi_{1j}^N e_{1j}^N(s),$$

$t \backslash \hat{u}_1$	$\hat{u}_1^4(t)$	$\hat{u}_1^8(t)$	$\hat{u}_1^{16}(t)$	$\bar{u}_1(t)$
0	-1.0602	-1.0599	-1.0598	-1.0598
0.25	-0.8419	-0.8419	-0.8419	-0.8419
0.5	-0.6209	-0.6241	-0.6239	-0.6239
0.75	-0.4008	-0.4030	-0.4060	-0.4060
1.0	-0.2268	-0.2116	-0.2022	-0.1880
1.25	-0.1743	-0.1860	-0.1884	
1.5	-0.1897	-0.1878	-0.1880	
1.75	-0.1877	-0.1880	-0.1880	
2.0	-0.1880	-0.1880	-0.1880	-0.1880
$t \backslash \hat{u}_2$	$\hat{u}_2^4(t)$	$\hat{u}_2^8(t)$	$\hat{u}_2^{16}(t)$	$\bar{u}_2(t)$
0	-0.8721	-0.8719	-0.8718	-0.8718
0.25	-0.8721			
0.5	-0.8721			
0.75	-0.8721	-0.8719		
1.0	-0.8720	-0.8718		
1.25	-0.8720			
1.5	-0.8719			
1.75	-0.8718			
2.0	-0.8719	-0.8718	-0.8718	-0.8718
$J(\hat{u})$	1.4018	1.4017	1.4017	1.4017

Table 4.1

\hat{x}_1 t	$\hat{x}_1^4(t)$	$\hat{x}_1^8(t)$	$\hat{x}_1^{16}(t)$	$\bar{x}_1(t)$
0.25	0.76221	0.76227	0.76228	0.76229
0.5	0.57927	0.57902	0.57905	0.57906
0.75	0.45179	0.45050	0.45031	0.45032
1.0	0.37519	0.37582	0.37500	0.37607
1.25	0.32780	0.32910	0.32905	0.32906
1.5	0.28219	0.28203	0.28205	0.28208
1.75	0.23498	0.23503	0.23504	0.23504
2.0	0.18799	0.18802	0.18803	0.18803

\hat{x}_2 t	$\hat{x}_2^4(t)$	$\hat{x}_2^8(t)$	$\hat{x}_2^{16}(t)$	$\bar{x}_2(t)$
0.25	1.03198	1.03203	1.03205	1.03205
0.5	1.06395	1.06407	1.06409	1.06410
0.75	1.09593	1.09610	1.09614	1.09615
1.0	1.12791	1.12814	1.12818	1.12821
1.25	1.12906	1.12933	1.12934	1.12941
1.5	1.07761	1.07790	1.07797	1.07799
1.75	0.98737	0.98748	0.98755	0.98758
2.0	0.87187	0.87180	0.87179	0.87180

Table 4.2

which together with Π_{00}^N determines the feedback law of the Nth approximation. At the mesh-points, we simply have

$$\Pi_1^N(0) = \Pi_{11}^N + \Pi_{12}^N, \quad \Pi_1^N(t_{1j}^N) = \Pi_{1,2j-1}^N - \Pi_{1,2j}^N, \quad j = 1, \dots, N.$$

Example 4.2. This is the problem of minimizing

$$J(u) = \int_0^{\infty} [x(t)^2 + u(t)^2] dt$$

subject to

$$\dot{x}(t) = x(t) + x(t-1) + u(t), \quad t \geq 0,$$

$$x(0) = 0, \quad x_0(t) = \sin \pi t, \quad -1 \leq t \leq 0.$$

Table 4.3 gives the optimal costs $J^N = \langle (x(0), x_0), \Pi^N(x(0), x_0) \rangle_{M^2}$ of the approximating systems and the costs $J(\hat{u}^N)$ when the original system is controlled by \hat{u}^N .

N	J^N	$J(\hat{u}^N)$
4	0.32117	0.32143
8	0.32138	0.32143
16	0.32142	0.32143

Table 4.3

Table 4.4 presents the feedback gains Π_{00}^N and $\Pi_1^N(s)$ at the meshpoints $-\frac{j}{4}$, $j = 0, \dots, 4$.

	Π_{00}^4	Π_{00}^8	Π_{00}^{16}
	2.8083	2.8092	2.8094
j	$\Pi_1^4(-\frac{j}{4})$	$\Pi_1^8(-\frac{j}{4})$	$\Pi_1^{16}(-\frac{j}{4})$
0	0.6349	0.6365	0.6369
1	0.8700	0.8801	0.8838
2	1.2544	1.2745	1.2803
3	1.8518	1.8812	1.8895
4	2.7507	2.7930	2.8050

Table 4.4

In Table 4.5, we list the values of $\hat{u}^N(\frac{j}{4})$, $\hat{x}^N(\frac{j}{4})$, $j = 0, \dots, 12$.

\hat{u} t	$\hat{u}^4(t)$	$\hat{u}^8(t)$	$\hat{u}^{16}(t)$
0.	0.86968	0.86836	0.86818
0.25	0.64873	0.64888	0.64891
0.5	0.49572	0.49647	0.49657
0.75	0.36350	0.36395	0.36400
1.0	0.24664	0.24624	0.24618
1.25	0.16200	0.16153	0.16147
1.5	0.11027	0.10997	0.10993
1.75	0.08030	0.08022	0.08021
2.0	0.06013	0.06015	0.06015
2.25	0.04349	0.04347	0.04347
2.5	0.02988	0.02982	0.02982
2.75	0.02002	0.01996	0.01995
3.0	0.01377	0.01373	0.01372

\hat{x} t	$\hat{x}^4(t)$	$\hat{x}^8(t)$	$\hat{x}^{16}(t)$
0.25	0.11276	0.11260	0.11258
0.5	0.05337	0.05332	0.05332
0.75	-0.06642	-0.06628	-0.06626
1.0	-0.10870	-0.10849	-0.10846
1.25	-0.06170	-0.06160	-0.06158
1.5	-0.01396	-0.01396	-0.01397
1.75	0.00758	0.00753	0.00752
2.0	0.00180	0.00178	0.00178
2.25	-0.00787	-0.00784	-0.00784
2.5	-0.01034	-0.01030	-0.01029
2.75	-0.00648	-0.00646	-0.00646
3.0	-0.00178	-0.00178	-0.00178

Table 4.5

Example 4.3. A simplified model for a wind tunnel at NASA's Langley Research Center is given by (see [4])

$$(4.36) \quad \dot{x}(t) = \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & -2\xi\omega \end{bmatrix} x(t) + \begin{bmatrix} 0 & ka & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t - 0.33) + \begin{bmatrix} 0 \\ 0 \\ -\omega^2 \end{bmatrix} u(t), \quad t \geq 0,$$

$x(0) = \text{col}(-0.1, 8.547, 0) \equiv x_0(t)$, $-0.33 \leq t \leq 0$, where $k = -0.0117$, $\xi = 0.8$, $\omega = 6.0$, $\frac{1}{a} = 1.964$. One wants to minimize

$$J(u) = \int_0^{\infty} [10^4 x_1(t)^2 + u(t)^2] dt$$

subject to (4.36).

The true solution of the problem was given in [16]. Note that the matrix W_0 weighting the contribution of the state trajectory to the costs is singular in this example, in contrast to the assumptions in Theorem 3.5. However, the piecewise linear approximation scheme produced the following values for $J(\hat{u}^N)$ and J^N .

N	$J(\hat{u}^N)$	J^N
4	136.4490	136.1785
8	136.4490	136.2921
16	136.4493	136.3486
$J(u)$	136.4049	

Table 4.6

N	Π_{00}^N		
4	8677.02161	-9.81498	-0.94768
	-9.81498	0.01850	0.00186
	-0.94768	0.00186	0.00019
8	8677.02502	-9.81503	-0.94768
	-9.81503	0.01851	0.00186
	-0.94768	0.00186	0.00019
16	8677.02551	-9.81504	-0.94768
	-9.81504	0.01851	0.00186
	-0.94768	0.00186	0.00019
Π_{00}	8677.02405	-9.81505	-0.94768
	-9.81505	0.01851	0.00186
	-0.94768	0.00186	0.00019

Table 4.7

In Table 4.7, we compare the first block of the Riccati matrix Π^N with the \mathbb{R}^3 -component of Π . The matrices $\Pi_1(t)$ and $\Pi_1^N(t)$,

$-h = -0.33 \leq t \leq 0$, have nonzero entries only in their second columns, which are shown in Table 4.8 for $t = -\frac{jh}{4}$, $j = 0, \dots, 4$.

j	0	1	2	3	4
$\Pi_1^4(-\frac{jh}{4})$	-41.39647 0.06915 0.00669	-43.83755 0.06653 0.00641	-46.36726 0.06358 0.00614	-48.97855 0.06095 0.00589	-51.67634 0.05845 0.00564
$\Pi_1^8(-\frac{jh}{4})$	-41.39710 0.06917 0.00668	-43.84694 0.06633 0.00640	-46.37700 0.06359 0.00614	-48.98892 0.06097 0.00589	-51.68730 0.05847 0.00565
$\Pi_1^{16}(-\frac{jh}{4})$	-41.39721 0.06917 0.00668	-43.84929 0.06631 0.00640	-46.37952 0.06360 0.00614	-48.99157 0.06098 0.00589	-51.69010 0.05847 0.00565
$\Pi_1(-\frac{jh}{4})$	-41.39721 0.06917 0.00668	-43.85008 0.06632 0.00641	-46.38034 0.06360 0.00614	-48.99246 0.06098 0.00589	-51.69103 0.05847 0.00565

Table 4.8

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